GENERAL LIBRARY
AUG 18 1921
UNIV. OF MICH.

VOLUME XVIII SECOND SERIES NUMBER 1

THE PHYSICAL REVIEW

A JOURNAL OF EXPERIMENTAL AND THEORETICAL PHYSICS

CONDUCTED BY

THE AMERICAN PHYSICAL SOCIETY

BOARD OF EDITORS

FREDERICK BEDELL, Managing Editor

HENRY CREW	E. L. NICHOLS	G. S. FULCHER
L. V. KING	C. M. SPARROW	F. A. SAUNDERS
H. S. UHLER	W. F. G. SWANN	O. M. STEWART

JULY, 1921

The Principles of Projective Covariance. ISRAEL HALLISH 1
The Laws of Absorption of X-Rays. F. K. RICHTMYER 13
The Effect of a Uniform Magnetic Field on the Motion of Electrons between Coaxial Cylinders. ALBERT W. HULL 31
Theory of the Motion of Electrons between Coaxial Cylinders taking into Account the Variation of Mass with Velocity. LEIGH PAGE 58
The Transmission of Waves through a Symmetric Optical Instrument. IRWIN ROMAN 60
An Extension of the Range of the McLeod Gauge. A. E. PFUND 76
New Books. 80

PUBLISHED BY
THE AMERICAN PHYSICAL SOCIETY
LANCASTER, PA., AND ITHACA, N. Y.

Entered at the post-office, at Lancaster, Pa. as second class matter

THE AMERICAN PHYSICAL SOCIETY

OFFICERS OF THE SOCIETY

President	THEODORE LYMAN	Harvard University, Cambridge, Mass.
Vice-President	C. E. MENDENHALL	Univ. of Wisconsin, Madison, Wis.
Secretary	D. C. MILLER	Case School of Applied Science, Cleveland, Ohio
Treasurer	G. B. PEGRAM	Columbia University, New York City
Editor	F. BEDELL	Cornell University, Ithaca, N. Y.
Assistant Secretary for the Pacific Coast	E. P. LEWIS	University of California, Berkeley, Cal.

The Council of the Society consists of the president, vice-president, secretary, treasurer, manager and editor, all living past presidents and eight elected members as follows:

Past Presidents: A. A. Michelson, A. G. Webster, Carl Barus, E. L. Nichols, Henry Crew, W. F. Magie, Ernest Merritt, R. A. Millikan, J. S. Ames.

Elected Members: A. L. Day, G. F. Hull, G. K. Burgess, J. C. McLennan, F. B. Jewett, Max Mason, K. T. Compton, H. M. Randall.

A complete list of past officers is given on a back page of the REVIEW.

THE PHYSICAL REVIEW

Published Monthly — Two Volumes Annually

Annual Subscription, throughout the Postal Union, $7.00, in advance; Single Copies, $.75.

Annual Index 1893–1920 (Issued June, 1921) $4.00, postage extra. Back Volumes (either series) $4.50 per volume, carriage extra. Back Numbers (either series) $.75 a copy, postage extra. Discount to subscription agencies only, ten per cent.

NOTE.—When remittance accompanies the order, the Index will be sent prepaid for $4.00 and single back numbers will be sent prepaid for $.75. Back numbers will only be sent when they can be furnished without breaking a volume, and back volumes when they can be furnished without breaking a continuous series. Orders that can not be completely filled will be filled so far as stock permits, unless otherwise specified in the order. Money will be refunded for any defective copies returned. No bound volumes can be supplied.

THE REVIEW offers to purchase, at $1.50 a copy, a limited number of No. 1, Vol. I, *First Series* and No. 2, Vol. IX, February 1917, *Second Series*.

Membership certificates will be furnished with bills, when so ordered, for twenty-five cents each; copies of bills, on forms furnished by customer, will be furnished for five cents each. Stamps will be accepted for these.

Members of the Physical Society may obtain back numbers at a discount of 10 per cent, provided statement of membership accompanies order. When a member's name has been dropped from the mailing list of the REVIEW on account of non-payment of dues, it will be restored to the list when dues are paid; any numbers that he has failed to receive in the interim will be furnished without cost upon application to the REVIEW. *All complaints for failure to receive the Review should be addressed to the Review and not to any officer of the Society.*

First Series. From July 1, 1893, to January 1, 1913, the REVIEW was conducted by Edward L. Nichols, Ernest Merritt and Frederick Bedell, with the financial support of Cornell University. During this period thirty-five volumes were issued, constituting the First Series. The first four volumes were issued annually; subsequent volumes, semi-annually.

Second Series. On January first, 1913, the REVIEW was transferred to the American Physical Society. A second series was then begun, the January, 1913, number being designated Volume I, Number 1, Second Series. (An announcement of this transfer will be found on the first page of the new series.) Semi-annual volumes begin in January and July.

For advertising rates, address THE PHYSICAL REVIEW, *Ithaca, N. Y.* Change of copy in standing advertisements should be sent direct to the printers, THE NEW ERA PRINTING CO. *Lancaster, Pa. Old copy is used until new copy is received.*

Subscribers to the REVIEW are referred to page two of the advertisements for further information. Permission is given to republish any article appearing in the REVIEW, provided proper credit is given.

Address Correspondence to

THE PHYSICAL REVIEW, ITHACA, N. Y.

CONTENTS OF VOL. XVIII., SECOND SERIES

JULY, 1921

The Principles of Projective Covariance. ISRAEL MAIZLISH 1
The Laws of Absorption of X-Rays. F. K. RICHTMYER 13
The Effect of a Uniform Magnetic Field on the Motion of Electrons between Coaxial Cylinders. ALBERT W. HULL 31
Theory of the Motion of Electrons between Coaxial Cylinders taking into Account the Variation of Mass with Velocity. LEIGH PAGE 58
The Transmission of Waves through a Symmetric Optical Instrument. IRWIN ROMAN. 62
An Extension of the Range of the McLeod Gauge. A. H. PFUND 78
New Books 82

AUGUST, 1921

Proceedings of the American Physical Society 85

Minutes of the One Hundred and Ninth Meeting. Crystal Structure of Titanium, Zirconium, Cerium, Thorium and Osmium, *Albert W. Hall;* The Width of X-Ray Spectral Lines, *L. Gilchrist;* Characteristic Low Voltage X-Radiation from Arcs in Metallic Vapors, *F. L. Mohler and Paul D. Foote;* Secondary High Frequency Radiation, *A. H. Compton;* Polarization of Secondary X-Rays, *A. H. Compton and C. F. Hagenow;* A Remeasurement of the Radiation Constant, by Means of X-Rays, *William Duane, H. H. Palmer and Chi-Sun Yeh;* Soft X-Rays of Characteristic Type, *E. H. Kurth;* The Change of the Mobility of the Positive Ions in Air with Age, *Henry A. Erikson;* The Absolute Sizes of Certain Monovalent Ions, *Wheeler P. Davey;* Forces Within a Static Atom, *Irving Langmuir;* Effect of Roughening of Surface on the Dispersion of Cannon Ball in Falling Through Air, *F. C. Brown;* On The Determination of Instantaneous Pressures in Guns by Means of a Piezo-Electric Gage and a Ballistic Galvanometer, *J. C. Karcher;* A New General Law of Deformation, *P. G. Nutting;* Contractions and Expansions of Amalgams with Time, *Arthur W. Gray;* Measurement of Mercury Vapor Pressure from 0° to 35° by the Knudsen Pressure Gauge, *C. F. Hill;* Thermal Conductivity of White Marble and "Neat" Cement, *Roy A. Nelson;* The Effect of Pressure on the Thermal Conductivity of Metals, *P. W. Bridgman;* The New Fixed Point on the Thermometric Scale, *Frederick Bates and F. P. Phelps;* A Disappearing Filament Optical Pyrometer Free from Diffraction Effects at the Filament, *C. O. Fairchild;* The Organ Pipe as a Coupled System, *Arthur Taber Jones;* On the Measurement of the Acoustic Output and Efficiency of Fog-alarm Apparatus, *Louis V. King;* A Simple Apparatus for Testing Pitch Control, *Preston Edwards;* Some New Consequences of the Theory of Ultimate Rational Units, *Gilbert N. Lewis;* Present Status of the Electric and Magnetic Units, *J. H. Dellinger;* Discussion of a Kinetic Theory of Gravitation, II., and Some New Experiments in Gravitation, *Charles F. Brush;* The Transmission of Waves through a Symmetric Optical Instrument, *Irwin Roman;* A Direct Reading Spectro-photometer for Measuring the Transmissivity of Liquids, *Irwin G. Priest;* The Excitation of Enhanced Spectra in Low Voltage Arcs, *Paul D. Foote, W. F. Meggers and F. L. Mohler;* On the Photoelectric Effect of Alkali Vapors, *Jakob Kunz;* Photo-Electric Potentials from the Retina, *E. L. Chaffee and W. T. Bovie;* The Variation of Metallic Conductivity with Electrostatic Charge, *Henry A. Perkins;* The Distributed Capacity

403933

of Inductance Coils, *G. Breit;* Some Effects of the Distributed Capacity of Coils to Ground, *G. Breit;* The High-Frequency Resistance of Inductance Coils, *G. Breit;* A Method of Measuring Coil Capacities and Standardizing Wavemeters, *G. Breit;* Tables for the Calculation of the Inductance of Circular Coils of Rectangular Cross Section, *Frederick W. Grover;* On Some New Formulæ for the Direct Numerical Calculation of the Coefficient of Mutual Induction of Coaxial Circles, *Louis V. King;* Note on the Earth's Magnetic Field, *W. F. G. Swann;* The Electrical Resistance of the Human Body, *N. Forman, J. S. Martin and F. Wenner;* New Methods for Maintaining Constant Frequency in High-frequency Circuits, *W. G. Cady;* Magnetic Rotatory Dispersion in Transparent Liquids, *R. A. Castleman, Jr., and E. O. Hulburt;* Total Emissive Powers and Resistivities of Tungsten at Incandescence, *A. G. Worthing and W. E. Forsythe;* Color Temperature of High Efficiency Lamps, *W. E. Forsythe;* the Detecting Efficiency of the Resistance-Capacity Coupled Electron Tube Amplifier, *E. O. Hulburt;* Radio Signal Fading Phenomena, *J. H. Dellinger and L. E. Whittemore;* Some Physical Problems of Aircraft Radio, *L. E. Whittemore;* The Design of a Standard Wavemeter, *R. T. Cox;* The High-Frequency Impedance of Radio Telephone Receivers, *C. T. Zahn;* A Recorder of Radio Time Signals, *E. A. Eckhardt and J. C. Karcher;* An International Series of Radio Audibility and Directional Measurements, *Carl Kinsley;* Detection Coefficient of Thermionic Valves, *E. L. Chaffee;* The Radio Research Field, *J. H. Dellinger and L. E. Whittemore;* On Emergent β Rays from Materials of Different Atomic Weight and Different Thickness Produced by γ Rays; Studies by the Statistical Method, *Alois F. Kovarik;* High Frequency Limits of X-Ray Spectra at Different Angles with the Cathode Stream, *D. L. Webster;* A Generalization of the Electrodynamic Equations with Applications to the Structure of the Electron and Radiationless Orbits, *Leigh Page;* Resistance and Inductance of a Three-Phase Cable, *Harvey L. Curtis;* Thickness of Stratified Soap Films, *P. V. Wells;* Temperature of an Electrically Heated Filament, *Chester Snow;* Standard Wave Lengths and the Constant Frequency Differences in the Spectra of Inert Gases, *W. F. Meggers;* Note on the Diurnal Variation of the Atmospheric Electric Potential Gradient, *S. J. Mauchly;* Peculiar Ice Formations, *H. G. Dorsey;* On the Exploded Wire Spectrum of Calcium, *R. A. Sawyer and A. E. Becker;* On the Radiation of Energy from Coils, *A. G. Webster;* On the Vibration of Gun Barrels, *A. G. Webster.*

SEPTEMBER, 1921

The Detecting Efficiency of the Resistance Capacity Coupled Electron Tube Amplifier. E. O. HULBURT 165

An Acoustical Galvanometer for the Measurement of Small Alternating Currents. F. R. WATSON and L. B. HAM 178

Evidence for the Existence of Homogeneous Groups of Large Ions. P. J. NOLAN 185

Thermoelectric Electric Effects in Iron and Mercury due to Asymmetric Heating. J. M. BENADE 199

Note on the Dynamics of Capillary Flow. WILLARD GARDNER 206

The Crystalline Structures of Sputtered and Evaporated Metallic Films. H. KAHLER .. 210

On the Emission and Absorption Wave-Lengths of the Characteristic Radiation in the L-Series. D. COSTER 218

Physical Properties of Thin Metallic Films. III. Some Factors Affecting the Resistance of Sputtered Platinum Films. LEWIS R. KOLLER 221

The Distinction between Intrinsic and Spurious Contact E.M.F.s and the Question of the Absorption of Radiation by Metals in Quanta. R. A. MILLIKAN 236

X-Ray Analysis of Three Series of Alloys. MARY R. ANDREWS 245

OCTOBER, 1921

Physical Properties of Thin Metallic Films. L. F. CURTISS 255
Conductivity of Flames Containing Salt Vapors. A. B. BRYAN 275
Generalization of Electrodynamics with Applications to the Structure of the Electron and to non-radiating Orbits. LEIGH PAGE 292
An Electric Doublet Theory of the Nature of the Molecular Forces of Chemical and Physical Interaction. R. D. KLEEMAN 303
Proceedings of the American Physical Society 313

Minutes of the One Hundred Tenth Meeting; The Moon's Electric Charge, *Fernando Sanford;* Earth Currents and the Sun's Induction, *Fernando Sanford;* The Sun's Charge: A Criticism of Methods, *F. J. Rogers;* The Effect of Temperature on the 3883 CN Band, *Raymond T. Birge;* Some X-Ray Isochromats, *David L. Webster;* The Width of X-Ray Spectrum Lines, *Arthur H. Compton;* The Effect of Damping on the Width of X-Ray Spectrum Lines, *G. E. M. Jauncey;* Experiments on Polarization of X-Rays, *Paul Kirkpatrick;* An X-Ray Spectrometer for the Determination of Absorption Coefficients, *Elmer Dershem;* A Photographic Method of X-Ray Crystal Analysis, *Elmer Dershem;* Secret Photo-Electric Signalling, *Elmer Dershem;* An Harmonic Curve Writer, *William J. Raymond;* The Variation of the Thermal Conductivity of Soft Cast Iron with the Temperature, *Elmer E. Hall, O. R. Hull and O. S. Imhoff;* The Index of Refraction of Water, Ethyl Alcohol, and Carbon Bisulphide at Various Temperatures, *Arthur R. Payne and Elmer E. Hall;* Preliminary Report on Spectroscopic Investigations in the Extreme Ultra-Violet, *J. J. Hopfield;* The Effect of Pressure on the Band Spectra of Nitrogen, *Jean Huddleston;* The Reaction of Iron with Nitric Acid, *Joseph G. Brown;* A Braun Tube, Undamped Wave, Precision Method of Determining Dielectric Constants of Gases, *L. T. Jones and H. G. Tasker;* The Effect of Ultra-Violet Light on the Viscosity of Rubber Colloids, *Richard Hamer;* The Simplest Mercury Vapor Pump, *L. T. Jones;* A New Rotary Mercury Pump, *L. T. Jones;* Atomic Radii, *Maurice L. Huggins;* Structure of the L-Series of Tungsten and Platinum, *Frank C. Hoyt;* Recent Observations of Absorption Spectra, *Arthur S. King;* Preliminary Measurements of the Critical Potentials of the M Lines in the X-Ray Spectrum of Lead, *P. A. Ross:* A Possible Origin of the Defect of the Combination Principle in X-Rays, *Arthur H. Compton.*

NOVEMBER, 1921

The Infra-red Absorption Spectra of Alkali Hydroxides. G. E. GRANTHAM 339
An Absolute Scale of X-Ray Wave Lengths. OSWALD B. OVERN 350
Thermal Conductivity of Some Wearing Materials. EMILY S. ROOD 356
Vapor Pressure of Metallic Calcium. NORMAN B. PILLING 362
Light Emission from a Moving Source in Connection with the Relativity Theory. N. RASHEVSKY 369
Note on the Plane Wave in an Isotropic Dielectric. SATYENDRA RAY 377
Note on Radiation Pressure. R. N. GHOSH AND N. K. SUR 380
The Ionization Potential of Selenium Vapor. A. D. UDDEN 385
On Talbot's Bands and the Theory of the Lummer-Gehrcke Interferometer. NIHAL KARAN SETHI 389
The Reflection of X-Rays by Crystals. H. A. WILSON 396
The Effect of the Age and the Concentration of a Solution of Rhodamine-B upon the Photoelectric Current Developed. RUTH YEATON JENKINS 402
Temperature Effect in Barium and Strontium Photo-Electric Cells. THEODORE W. CASE 413

DECEMBER, 1921

Positive Ray Analysis of Lithium and Magnesium. A. J. DEMPSTER 415
Duration of Impact of Bars. IRWIN W. TSCHUDI 423
The Binaural Location of Pure Tones. R. V. L. HARTLEY AND THORNTON C. FRY 431
A Comparison of the Thermionic and Photo-Electric Work Function for Platinum. OTTO KOPPIUS 443
Some Facts Bearing on the Structure of Atoms, Particularly of the Helium Atom. R. A. MILLIKAN 456
On the Extension of the X-Ray Spectrum. E. H. KURTH 461
Addenda and Errata 477
Index 481

Second Series *July, 1921* *Vol. XVIII, No. 1*

THE

PHYSICAL REVIEW.

THE PRINCIPLE OF PROJECTIVE COVARIANCE.

BY ISRAEL MAIZLISH.

SYNOPSIS.

The Principle of Projective Covariance.—The principle here presented asserts that the laws of physics are covariant with respect to the following transformation: $x_c = G(\sigma, r)x$, $y_c = G(\sigma, r)y$, $z_c = G(\sigma, r)z$, $t_c = G(\sigma, r)t$, where x, y, z, t are the Minkowski variables, r is the distance of a point in S_4 from the origin of a system of coördinates, and σ is an arbitrary constant.

It is shown that if Ψ is a physical quantity depending only upon a single variable ξ, the relation between Ψ and ξ is $\Psi = K\xi^{\alpha/\beta}$, where $\Psi_c = G^{\alpha}(\sigma, r)\Psi$, $\xi_c = G^{\beta}(\sigma, r)\xi$, and K = constant. Examples are given of the application of the principle.

Comparison with the Principle of Similitude.—While there are similarities between the present work and Dr. Tolman's, the Principle of Projective Covariance does not in any way depend on the Principle of Similitude. It is pointed out that the Principle of Similitude, however, may be regarded as a very special interpretation of the Principle here presented. Concepts like "miniature universe" are shown to be extraneous, and, by taking a special form of the Principle of Projective Covariance, it is shown that the latter may be so interpreted as to give us also an extended universe.

1. While endeavoring to discover a simple method of solving certain problems in mathematical physics, especially the one relating to blackbody radiation, it occurred to the writer that there might exist a transformation with respect to which the laws of physics were covariant. A subsequent investigation showed that, as far as it has been carried, this is the case.

2. Considering a space of four dimensions (S_4) we will now make the following

Postulate: The laws of physics are covariant with respect to the following transformation:

$$\begin{aligned} x_c &= G(\sigma, r)x, \\ y_c &= G(\sigma, r)y, \\ z_c &= G(\sigma, r)z, \\ t_c &= G(\sigma, r)t, \end{aligned} \qquad (C)$$

where x, y, z, and t are the Minkowski variables, and $G(\sigma, r)$ is an arbitrary[1] function of the distance, r, and an arbitrary constant[2] σ.

This Postulate, for the sake of brevity, will be called the *Principle of Projective Covariance.*[3]

SOME TRANSFORMATION-EQUATIONS.

3. We will now proceed to derive a number of equations connecting certain physical quantities with those obtained by the application of the Principle of Projective Covariance.

4. *Distance:* Let $P(x, y, z, t)$ be a point in S_4 whose distance from the origin of a system of coördinates is r. Then,[4] if $P_c(x_c, y_c, z_c, t_c)$, and r_c be the corresponding quantities when the transformation (C) is applied, it is readily seen that

$$r_c = G(\sigma, r)r. \tag{1}$$

5. *Velocity:* To derive the transformation-equation for velocity we note that, by definition, the dimensions of velocity are

$$[v] = [L][T^{-1}].$$

By the Principle of Projective Covariance,

$$[v_c] = [L_c][T_c^{-1}].$$

But

$$l_c = G(\sigma, r)l,$$

and

$$t_c = G(\sigma, r)t.$$

Therefore,

$$v_c = v. \tag{2}$$

6. *Acceleration:* The dimensions of acceleration are

$$[a] = [L][T^{-2}].$$

Since

$$l_c = G(\sigma, r)l,$$

and

$$t_c = G(\sigma, r)t,$$

[1] It is possible to give the function $G(\sigma, r)$ such a form as to make it zero, infinite, or indeterminate for a certain value of r. The transformation would then be meaningless. There is, however, no particular reason for imputing such a form to $G(\sigma, r)$ rather than one of a multiplicity of legitimate functions.

[2] It is convenient to include σ in the function $G(\sigma, r)$. We shall presently consider a special form of $G(\sigma, r)$ in order to manifest conspicuously the differences between the present work and Dr. Tomlan's.

[3] The word *projective* is used here to mean non-metrical. The title is suggested by the special form of $G(\sigma, r)$ considered in paragraph 29. There, it may be added, the word has its usual significance.

[4] The subscript $_c$ will be used to signify that the quantity in question has been obtained by the application of the Principle of Projective Covariance.

it is seen that[1]

$$a_c = G^{-1}(\sigma, r)a. \tag{3}$$

7. *Force:* Universal or proportionality constants will obviously be unaffected by the Principle of Projective Covariance. Further, we may regard the number of Faraday tubes having their ends on a body as a measure of the charge on that body. The unit of electricity is, accordingly, the Faraday tube. Let us now consider a conductor C. The number of Faraday tubes having their ends on C is equal to the charge, q, on C. The only way in which our transformation-equations can affect the number of these tubes is to alter their "density" upon C, either by a uniform compression or expansion, or, in general, by a distortion of C. But the *total number* of Faraday tubes will remain unaltered.[2] The charge, q, will therefore be the same as it was originally. It is thus evident that q is invariant under the above transformation; that is,

$$q_c = q. \tag{4}$$

Now, Coulomb's Law asserts that the force between two electric charges, q and q', is directly proportional to their product, and inversely to the square of the distance between them; that is,

$$F \propto \frac{q \cdot q'}{r^2}. \tag{a}$$

By the Principle of Projective Covariance we also have

$$F_c \propto \frac{q_c \cdot q_c'}{r_c^2}. \tag{b}$$

Divide (*b*) by (*a*). The result is:

$$F_c = \left[\frac{q_c \cdot q_c'}{q \cdot q'} \cdot \frac{r^2}{r_c^2}\right] F,$$

or, by virtue of (1) and (4),

$$F_c = G^{-2}(\sigma, r)F. \tag{5}$$

8. *Mass:* In general, however, we have

$$F = ms'' = (\text{mass})(\text{acceleration});$$

i.e.,

$$F = ma. \tag{f}$$

By the Principle of Projective Covariance,

$$F_c = m_c a_c.$$

[1] $G^n(\sigma, r)$ or simply G^n will be written for $[G(\sigma, r)]^n$.

[2] The consideration that the Faraday tubes may be modified by different "ethers" is trivial, as, in this paper, we are not concerned with model universes.

In consideration of (5) and (3) this equation may be written

$$G^{-2}(\sigma, r)F = m_c G^{-1}(\sigma, r)a.$$

Substituting ma for F, this reduces to

$$G^{-2}(\sigma, r)ma = m_c a \cdot G^{-1}(\sigma, r).$$

Hence,

$$m_c = G^{-1}(\sigma, r)m. \tag{6}$$

Temperature: To derive the transformation-equation for absolute temperature, we may proceed as follows:

9. According to the Boltzmann-Planck entropy-law, we may write

$$\Delta S = K \log \frac{W_2}{W_1},$$

where ΔS is the increase in entropy resulting from the passage of a body from state 1 to state 2. The ratio W_2/W_1 is easily seen to be a pure number, and does not contain σ. For, assuming with Gibbs[7] that

$$\log W = \frac{\Psi - \epsilon}{\theta},$$

a simple substitution shows that the dimension of W_2/W_1 is zero, and that σ does not appear in this ratio. Hence, on applying the transformation (C), we obtain

$$\Delta S_c = \Delta S.$$

But now,

$$\Delta S = \int \frac{dQ}{T} = \Delta S_c = \int \frac{dQ_c}{T_c}.$$

Hence, it follows that

$$\frac{dQ}{T} = \frac{dQ_c}{T_c}.$$

10. In this equation dQ has the dimensions of energy. If we let $E = mv^2/2$, and $E_c = m_c v_c^2/2$, it is readily seen that, in consideration of (2) and (6),

$$E_c = G^{-1}(\sigma, r)E. \tag{7}$$

The equation

$$\frac{dQ}{T} = \frac{dQ_c}{T_c}$$

now becomes

$$\frac{dQ}{T} = \frac{G^{-1}(\sigma, r) \cdot dQ}{T_c}.$$

Hence,

$$T_c = G^{-1}(\sigma, r)T. \tag{8}$$

[7] Gibbs, Statistical Mechanics.

11. The same result may be arrived at by a somewhat different procedure. Thus, according to Stefan's Law,

$$\varphi = \text{energy density} = \text{energy/volume},$$
$$= kT^4,$$

where k is a constant. By the Principle of Projective Covariance, and equations (1) and (7), it is seen that

$$\varphi_c = kT_c^4 = \frac{\varphi}{G^4(\sigma, \tau)} = \frac{kT^4}{G^4(\sigma, \tau)}.$$

Therefore,

$$T_c = G^{-1}(\sigma, \tau)T \tag{8}$$

as above.

TABLE OF TRANSFORMATIONS.

12. For the sake of future reference it will be convenient to tabulate the results already obtained.

Name of Quantity.	Transformation-Equation.
Length	$l_c = G(\sigma, \tau)l.$
Time	$t_c = G(\sigma, \tau)t.$
Velocity	$v_c = v.$
Acceleration	$a_c = G^{-1}(\sigma, \tau)a.$
Mass	$m_c = G^{-1}(\sigma, \tau)m.$
Force	$F_c = G^{-2}(\sigma, \tau)F.$
Pressure	$P_c = G^{-4}(\sigma, \tau)P.$
Density	$d_c = G^{-4}(\sigma, \tau)d.$
Volume	$V_c = G^3(\sigma, \tau)V.$
Energy	$E_c = G^{-1}(\sigma, \tau)E.$
Energy-density	$\varphi = G^{-4}(\sigma, \tau)\varphi.$
Temperature	$T_c = G^{-1}(\sigma, \tau)T.$
Frequency	$\nu_c = G^{-1}(\sigma, \tau)\nu.$

APPLICATION OF THE PRINCIPLE.

13. In order to obtain unique solutions of physical problems by means of the Principle of Projective Covariance, it has thus far been found necessary to restrict ourselves to the case where we have one independent variable (ξ), and one dependent variable (Ψ). It is for this case that a unique relation between Ψ and ξ has been found.

14. Let us suppose that some quantity Ψ is expressed as a function of some other quantity ξ; that is, that we have

$$\Psi = \varphi(\xi).$$

By the Principle of Projective Covariance we also have

$$\Psi_c = \varphi(\xi_c).$$

The transformation-equations for Ψ and ξ can easily be obtained. Let, then,

$$\Psi_c = G^n(\sigma, \tau)\Psi,$$

and

$$\xi_c = G^{\beta}(\sigma, r)\xi, \qquad (\alpha \neq 0, \quad \beta \neq 0), \tag{9}$$

where α and β are numerical constants. Then,

$$\Psi_c = G^{\alpha}(\sigma, r)\Psi = \varphi(\xi_c) = \varphi(G^{\beta}\xi);$$

and, therefore,

$$\Psi = \varphi(\xi) = G^{-\alpha}(\sigma, r)\varphi[G^{\beta}(\sigma, r)\xi]. \tag{10}$$

15. It will now be shown that the solution of the functional equation (10) is

$$\Psi = K\xi^{\alpha/\beta},$$

where K is a constant.

16. To this end, let

$$G^{-\alpha}(\sigma, r)\xi^{-\alpha/\beta}\varphi[G^{\beta}(\sigma, r)\xi] = \eta[G^{\beta}(\sigma, r)\xi]. \tag{11}$$

Next, combine this equation with equation (10). The result is

$$\xi^{-\alpha/\beta}\varphi(\xi) = \eta[G^{\beta}(\sigma, r)\xi]. \tag{12}$$

In eq. (12) replace ξ by $G^{\beta}(\sigma, r)\xi$. We shall then have

$$[G^{\beta}(\sigma, r)\xi]^{-\alpha/\beta}\varphi[G^{\beta}(\sigma, r)\xi] = \eta[G^{2\beta}(\sigma, r)\xi];$$

that is,

$$\eta[G^{\beta}(\sigma, r)\xi] = \eta[G^{2\beta}(\sigma, r)\xi]. \tag{13}$$

Now, by replacing ξ by $G^{-\beta}(\sigma, r)\xi$, equation (13) becomes

$$\eta[\xi] = \eta[G^{\beta}(\sigma, r)\xi]. \tag{14}$$

By virtue of equation (13) we also have

$$\eta[\xi] = \eta[G^{2\beta}(\sigma, r)\xi],$$

and, by mathematical induction,

$$\eta[\xi] = \eta[G^{n\beta}(\sigma, r)\xi]. \tag{15}$$

In equation (15) if $G^{n\beta}(\sigma, r) > 1$, let

$$\xi = G^{-n\beta}\zeta.$$

By this transformation, (15) will now become

$$\eta[\zeta] = \eta[G^{-n\beta}(\sigma, r)\zeta]. \tag{16}$$

From physical considerations it is clear that $\eta(0)$ exists. Hence, it follows, from (15) and (16), that the only solution of (14) is

$$\eta[\xi] = \eta[G^{\beta}(\sigma, r)\xi] = \text{const.} = K.$$

Therefore,

$$\xi^{-\alpha/\beta}\varphi(\xi) = \eta[G^{\beta}(\sigma, r)\xi] = K;$$

that is, since $\Psi = \varphi(\xi)$,

$$\Psi = K\xi^{\alpha/\beta}. \tag{A}$$

17. It follows that whenever we are certain that Ψ is a function of ξ only, all that is needed to get the relation between Ψ and ξ is to find α and β, and substitute the value of α/β in formula (A).

18. *Example 1:* As our first illustration, consider the photo-electric effect, and assume with Einstein that

$$\Psi = \Delta E = E - E_0 = \varphi(\nu),$$

where $\varphi(\nu)$ is an unknown function of the frequency, ν, of the incident light. By referring to the Table of Transformations it will be seen that

$$\alpha = -1, \quad \beta = -1, \quad \text{and} \quad \therefore \ \frac{\alpha}{\beta} = +1.$$

Hence, by formula (A),

$$\Delta E = h\nu,$$

where h is a constant (in this case, Planck's).

19. *Example 2:* It is known that the wave-length (λ_m) for which the spectral energy is a maximum is a function of the absolute temperature. Let us find the function. Assume that

$$\Psi = \lambda_m = \varphi(T).$$

By referring to the Table of Transformations it will be seen that

$$\frac{\alpha}{\beta} = -1.$$

Therefore, by formula (A),

$$\lambda_m = bT^{-1},$$

or,

$$\lambda_m T = b,$$

where b is a constant.

20. *Example 3:* As another illustration, consider the mean kinetic energy (E_m) of the molecules of a perfect gas. Let us assume that E_m is a function of the absolute temperature only. We thus have

$$\Psi = E_m = \varphi(T).$$

From the Table of Transformations it is seen that

$$\alpha = -1, \quad \text{and} \quad \beta = -1.$$

Hence, $\alpha/\beta = +1$, and, by formula (A),

$$E_m = nT,$$

where n is a constant.

21. It may here be remarked that the solutions of the four classical functional equations, namely,

$$\varphi(u+v) = \varphi(u) + \varphi(v), \tag{1}$$
$$\varphi(u+v) = \varphi(u)\varphi(v),$$
$$\varphi(uv) = \varphi(u) + \varphi(v),$$

and

$$\varphi(uv) = \varphi(u)\varphi(v),$$

have been obtained by means of formula (A). The solutions of the last three equations were made to depend on the solution of equation (1), the latter being readily obtained by the application of the Principle of Projective Covariance.

22. The Principle of Projective Covariance will help us to determine unknown functions which depend upon more than one variable. Consider, for example, the frequency, ν, of a thin, flexible string of density d, total mass m, radius r, and length l, when stretched by a force of F dynes. Let us assume that

$$\nu = \varphi(l, m, F). \tag{1}$$

By the Principle of Projective Covariance we also have

$$\nu_c = \varphi(l_c, m_c, F_c). \tag{2}$$

Referring to the Table of Transformations it is seen that

$$\nu_c = G^{-1}(\sigma, r)\nu, \qquad l_c = G(\sigma, r)l,$$

$$m_c = G^{-1}(\sigma, r)m, \quad \text{and} \quad F_c = G^{-2}(\sigma, r)F.$$

Therefore,

$$\nu_c = G^{-1}(\sigma, r)\nu = \varphi(l_c, m_c, F_c)$$

$$= \varphi(Gl, G^{-1}m, G^{-2}F)$$

and

$$\nu = \varphi(l, m, F) = G\varphi(Gl, G^{-1}m, G^{-2}F). \tag{3}$$

It is evident that one solution of (3) is

$$\nu = k\sqrt{\frac{F}{ml}},$$

where k is a constant. The general solution of (3), however, would be of the form

$$\nu = kl^{\alpha}m^{\beta}F^{\gamma},$$

where α, β, and γ are constants staisfying the following condition-equation:

$$\alpha - \beta - 2\gamma = -1. \tag{4}$$

23. To determine α, β, and γ uniquely, we would make two simple experiments, using the same material in both. In the first experiment we would vary F and keep m and l constant. In the second, we would vary m (that is, r) while keeping F and l fixed. The results of these

experiments[1] would be that

$$\gamma = \tfrac{1}{2}, \quad \text{and} \quad \beta = -\tfrac{1}{2}.$$

Using this information and the condition-equation (4), we get immediately,

$$\alpha = -\tfrac{1}{2}.$$

Hence,

$$\nu = k\sqrt{\frac{F}{ml}},$$

or, writing $m = \pi r^2 d \cdot l$,

$$\nu = k\sqrt{\frac{F}{\pi r^2 d l^2}}.$$

THE FUNDAMENTAL PROBLEM.

24. It should be observed that no direct method of getting a unique solution of a problem such as the one just given has yet been found. The Principle of Projective Covariance would, of course, be of very great service if it enabled us to solve uniquely the general equations

$$\begin{aligned} \Psi &= \varphi(\xi, \eta, \cdots, \omega), \\ \Psi_c &= \varphi(\xi_c, \eta_c, \cdots, \omega_c), \end{aligned} \tag{Q}$$

where

$$\Psi_c = G^\gamma(\sigma, r)\Psi, \qquad \xi_c = G^\alpha(\sigma, r)\xi,$$

$$\eta_c = G^\beta(\sigma, r)\eta, \qquad \cdots, \qquad \text{and} \qquad \omega_c = G^\delta(\sigma, r)\omega.$$

The solution of (Q) has not, however, been obtained. In default thereof, the Principle of Projective Covariance may still be used in verifying known functions (physical equations), and predicting possible forms of unknown functions.

25. In the case of two independent variables ξ and η, and if the reasonable assumption be made that Ψ is expressed as

$$\Psi = K\xi^n\eta^m,$$

the solution has been obtained in the form

$$\Psi = K\xi^n\eta^m, \tag{D}$$

with

$$n = (\gamma + \beta\epsilon)/(\alpha + \beta), \qquad m = (\gamma - \alpha\epsilon)/(\alpha + \beta),$$

$$\Psi_c = G^\gamma(\sigma, r)\Psi \qquad \xi_c = G^\alpha(\sigma, r)\xi,$$

$$\eta_c = G^\beta(\sigma, r)\eta, \qquad \text{and} \qquad \epsilon = n - m.$$

[1] These experiments are, of course, unnecessary, if we wish to combine with the Principle of Projective Covariance the theory of dimensions. In that case, α, β, and γ would be obtained by combining *two* equations from the theory of dimensions with the condition-equation given by the Principle of Projective Covariance.

26. Although the attempt to express ϵ in terms of α, β, and γ has not been successful, it is worth pointing out that formula (D), $[\alpha + \beta \neq 0$, $\gamma - \alpha\epsilon \neq 0$, and $\gamma + \beta\epsilon \neq 0]$, gives the correct result in every case.

27. At the time of writing it does not appear possible to solve (Q) with the help of the Principle of Projective Covariance only. Just how far the work can be extended, and what additional assumptions, in particular cases, will be needed, cannot be definitely stated without further investigation. It is hoped that there will be an opportunity, in the future, to consider anew some of the problems which originally led to the Principle here presented.

A Special Form of the Function $G(\sigma, r)$.

28. Since $G(\sigma, r)$ is arbitrary,[1] evidently $G(\sigma, r) = \sigma^a r^b$, where a and b are any constants, is a possible form of $G(\sigma, r)$. Now, upon placing a equal to 1, and b equal to zero, we get

$$G(\sigma, r) = \sigma.$$

29. The transformation with respect to which the laws of physics are covariant now takes the following special form:[2]

$$\left.\begin{aligned} x_c &= \sigma x, \\ y_c &= \sigma y, \\ z_c &= \sigma z, \\ t_c &= \sigma t, \end{aligned}\right\} \qquad (PT)$$

where σ is an arbitrary constant. From what has preceded, it is obvious that the transformation for length is

$$l_c = \sigma l.$$

The transformation-equations for the other quantities considered above are the same as those already given except that σ replaces $G(\sigma, r)$.

Comparison between the Present Work and Dr. Tolman's.

30. For the sake of comparison, the present work and Dr. Tolman's are presented in tabular form.

The Present Work.	*Dr. Tolman's Work.*
The Principle of Projective Covariance: The laws of physics are covariant with respect to the following transformation: $x_c = G(\sigma, r)x$, $y_c = G(\sigma, r)y$, $z_c = G(\sigma, r)z$, $t_c = G(\sigma, r)t$.	The Principle of Similitude: The fundamental entities of which the physical universe is constructed are of such a nature that from them a miniature universe could be constructed exactly similar in every respect to the present universe.

[1] See footnote 1.

[2] In this special form, the transformation is projective. Hence, the title *Projective Covariance.*

Transformations for fundamental physical quantities:

$$l_c = G(\sigma, r)l,$$
$$t_c = G(\sigma, r)t,$$
$$m_c = G^{-1}(\sigma, r)m,$$
$$q_c = q,$$
$$S_c = S.$$

Working formula for two variables:

$$\Psi = K\xi^{\alpha/\beta},$$

(see (A) above).

Relations between fundamental quantities of the miniature universe (denoted by primed letters) and those of the present universe:

$$m' = m/x, \qquad l' = xl,$$
$$q' = q, \qquad t' = xt, \qquad S' = S.$$

Working formula:

Not given by Tolman.

(See, however, foot-note 11.)

31. The reader will note that there are similarities between the present work and Dr. Tolman's: the method of development in some instances being almost identical in both papers. Nevertheless, in spite of these coincidences, the Principle of Projective Covariance is entirely independent of the Principle of Similitude. On the other hand, the Principle of Similitude may be regarded as a SPECIAL INTERPRETATION of the Principle of Projective Covariance.

32. The Principle of Similitude is a special case not only of the general Principle of Covariance, (C), but also of the special form of it considered in paragraph 29. For, since σ is entirely arbitrary, the transformation may be regarded either as a compression or an expansion. Hence, in addition to the "*miniature universe*" of the Principle of Similitude, the Principle of Projective Covariance (that is, the special form of it as given in paragraph 29), by a suitable interpretation, could also be made to yield what may be called an EXTENDED UNIVERSE.

33. The conclusion drawn from the preceding remarks is that concepts like "miniature universe" are extraneous. That this is the case, may be strikingly shown by considerations like the following:

34. Let us assume for the moment that the general function $G(\sigma, r)$ is of the form

$$G(\sigma, r) = n^{\sigma - r}, \qquad (a)$$

where n is any positive number greater than 1. The transformation for length is

$$r_c = G(\sigma, r)r. \qquad (b)$$

Now, in general, if $\sigma > r > 1$, it is evident from (a) and (b) that

$$r_c > r;$$

and if $r > \sigma > 1$, we shall have

$$r_c < r.$$

[11] Formula (A) can be shown to follow from the Principle of Similitude. This was done by the writer in a paper read before the Iowa Academy of Science, April, 1920.

Similar considerations hold for $\sigma > r < 1$, and $r > \sigma < 1$. The Principle of Projective Covariance holds whether $r_c > r$ or $r_c < r$. But, if we are contemplating a model universe, it would manifestly have to be of such a nature that, for certain values, a length would experience an expansion, while for other values, it would suffer a contraction. Hence, in general, a geometrical figure would experience a distortion; and, in certain cases, may not have any resemblance whatever to the original figure (that is, to the figure as seen by an observer of the present universe). Under these circumstances, then, the model universe would not be "*exactly similar in every respect to the present universe.*" [1]

35. In justice to Dr. Tolman it should be emphasized that none of his work is invalidated by the Principle here presented. It appears, however, that the Principle of Similitude is merely a very special case of the general Principle of Projective Covariance.

36. When one considers the concepts of space and time as reflected in the works of Einstein and Minkowski, and if one interprets the present Principle to mean that physical equations are "homogeneous in Minkowski variables," the simplicity of the Principle will be quite evident.

37. The chief advantages of the present work consist in (1) avoiding extraneous concepts (*e.g.*, "miniature universe") and the attendant unprofitable attempts at visualization; (2) not making use of Einstein's postulate of the constancy of the velocity of light; and (3) the derivation of a working formula.

I am grateful to Professor E. B. Wilson, of the Massachusetts Institute of Technology, for having read part of the manuscript.

REED COLLEGE,
PORTLAND, OREGON,
Dec., 1920.

[1] R. C. Tolman, PHYS. REV., Aug., 1914.

THE LAWS OF ABSORPTION OF X-RAYS.[1]

By F. K. Richtmyer.

Synopsis.

Measurement of Absorption of X-rays.—The use of a filtered beam of rays can at best give only approximate values of the coefficients, for such a beam is not homogeneous. It is pointed out that logarithmic absorption is no proof of homogeneity. In these experiments homogeneous radiation was secured by using a Bragg spectrometer and the effects of secondary radiation were avoided by placing the absorbing material between the two slits which limited the beam. By taking these and other precautions a precision of 1–2 per cent. was obtained. For wave-lengths below 0.15 Å. it was found better to place the thin analyzing crystal of NaCl nearly at right angles to the beam and to use the radiation reflected from atom planes perpendicular to the ones usually used.

X-ray Absorption by Ag, Al, Cu, Mo, Pb and H_2O for Wave-lengths about 0.13 to 0.7 Å.—The linear relation, established by Hull and Rice, between the *mass-absorption coefficient* and λ^3, $\mu/\rho = F\lambda^3 + \sigma/\rho$, holds accurately for these substances until, as λ is increased, the characteristic K limit is approached, when the values of μ/ρ are increasingly too small. The linear relation also holds above the K limit, F having a much smaller value but σ/ρ, the *mass-scattering coefficient* being about the same. The *atomic absorption coefficient* for wave-lengths below K is a linear function of N^4, where N is the atomic number, in agreement with Owen's results. The *general law of X-ray absorption* is therefore as follows: $\mu_a = kN^4\lambda^3 + \sigma_a$ where k is a universal constant equal to 2.29×10^{-27} below the k limit and to 0.33×10^{-27} above that limit. The experimental curves give no evidence for the existence of the "*J*" *radiation* reported by certain observers for Ag, Al, Cu and H_2O. The absorption for water was measured down to 0.093 A but shows no indication of an approach to the absorption found for the hard γ-rays of RaC.

General Statements on Absorption Measurements.

An accurate knowledge of the laws of X-ray absorption by various media is of obvious importance in connection with the problems of atomic structure and of the nature of radiant energy. One may classify all measurements of X-ray absorption coefficients according as the investigator (1) has used a strictly homogeneous beam obtained by reflection from a crystal grating or (2) has attempted to get a more or less homogeneous beam either by employing the characteristic radiation of some target or by filtering out the longer waves from a gross beam by a filter such as aluminum and then assuming that the resultant filtered beam is practically homogeneous. In the latter case the mean wave-

[1] The work herein described was done at the Research Laboratory of the General Electric Co. The writer wishes hereby to make grateful acknowledgment, particularly to Dr. A. W. Hull, for the many courtesies extended and for the laboratory facilities made available.

length of the beam is given. Thus Auren,[1] in studying the absorption coefficients of a large number of compounds, from which to compute the coefficients of the components, filters his radiation (voltage applied to tube not stated) through 1.12, 2.05, 5.00 and 10.00 mm. of aluminum respectively. He gets the mean wave-length of each filtered beam by measuring the absorption coefficient of copper for the beams and compares this coefficient with data from Barkla for the variation of the absorption coefficient of copper as a function of wave-length. The mean wave-length of the filtered beams are .38, .36, .34, and .30 Ångströms, respectively. He concludes, from absorption measurements by use of these wave-lengths, that "although the atomic absorption coefficient in a great number of elements may be proportional to a power of the atomic number (yet) . . . in certain elements there are very marked discontinuities."

In an earlier paper[2] Auren concludes that the atomic absorption coefficient is not proportional to the fourth power of the atomic number of the absorber, contrary to conclusions reached by Bragg and Pierce,[3] and confirmed later by Owen[4] for the absorption of the α line of Pd obtained by reflection from a crystal.

The use of a filtered beam for absorption measurements may give approximate results so long as one of the characteristic absorption limits of the absorber does not fall within the spectral region covered by the filtered beam; *but at best the results can be only approximations to the true values of the absorption coefficients.*

Another error should be pointed out in absorption measurements involving filtered radiation. Assuming the general absorption law for homogeneous radiation

$$I = I_0 e^{-\mu x}$$

one gets the linear equation between $\log I$ and x:

$$\log I = -\mu x + \log I_0.$$

Conversely it has been assumed[5] that a strictly homogeneous radiation has been obtained by filtering when a plot between $\log I$ and x is a straight line. This conclusion is not necessarily justified. The linear relation simply means that μ has the same value for all components of the radiation, which may be homogeneous or heterogeneous; or in other words that, for the radiation remaining after filtering, μ does not change

[1] Phil. Mag., 37, p. 165 (1919).
[2] Phil. Mag., 33, p. 471 (1917).
[3] Phil. Mag., 28, p. 626 (1914).
[4] Proc. Roy. Soc., 94, p. 510 (1919).
[5] *E.g.*, see Rutherford, Barnes, and Richardson, Phil. Mag., 30, p. 339 (1915).

with λ. As is shown by data contained herein and also by Barkla and White[1] there is comparatively little change in the mass absorption coefficient, μ/ρ, for water below $\lambda = .2$ Å. In the case of Al the change in μ/ρ from $\lambda = .15$ Å. down to the shortest X-ray wave-lengths yet obtained is not more than 20 per cent. This is the obvious explanation of the linear relation between log I and x after filtering through Al as obtained by Rutherford, Barnes, and Richardson, rather than the one given by these writers, namely that there is no decrease in wave-length in X-radiation by use of voltages above 125,000.

On the contrary absorption measurements in which a strictly homogeneous beam has been obtained by reflection from a crystal grating seem to lead to perfectly definite laws. Thus, Hull and Rice,[2] conclude that the mass absorption coefficient, μ/ρ, is given as a function of the wave-length, λ, by an equation of the form

$$\frac{\mu}{\rho} = F\lambda^3 + \frac{\sigma}{\rho}, \qquad (1)$$

where σ/ρ is the so-called mass-scattering coefficient, and F is a constant. These investigators verified this equation for aluminum, copper, and lead from .147 Å. to .392 Å. They assumed σ/ρ to be constant and equal to .12. In this equation the term $F\lambda^3$ is due to fluorescent radiation. It is frequently written τ/ρ, and is then called the mass-fluorescent absorption coefficient. Owen[5] shows that τ/ρ is proportional to the cube of the atomic number of the absorber, *i.e.*,

$$\frac{\tau}{\rho} = CN^3 \qquad (2)$$

at wave-length $\lambda = .589$ Å. But it should be remarked that Owen's results can hold only for those substances whose K absorption limit is on the long wave-length side of $\lambda = .589$ Å. A similar law may possibly hold for the group of elements whose K absorption limit is on the short wave-length side of $\lambda = .589$ Å., but with a different value of C.

Purpose of the Investigation.

In view of the simplicity of the laws expressed by (1) and (2) and of the corresponding importance which they thereby assume, it seemed desirable to extend these measurements over as great a range of wave-lengths as possible, getting measurements of the highest attainable precision, in order to answer the following questions.[3]

[1] Phil. Mag., 34, p. 270 (1915).

[2] Phys. Rev., VIII., p. 326 (1916).

[3] Preliminary reports of the results of this investigation have been given in papers presented to the American Physical Society. See Richtmyer and Grant, Phys. Rev., and

I. Do equations (1) and (2) accurately hold over a wide range of wave-lengths, especially if the measurements be extended to the long wave-length side of the K absorption limit—the latter of course with change in value of the constant F?

II. Is the term σ/ρ constant for all substances?

III. Is there, as reported by Barkla and White,[1] Williams,[2] Owen[3] and Dauvillier,[4] a discontinuity in the absorption curve, indicating a "J" radiation, similar to the discontinuities that mark the K and L radiations?

APPARATUS AND METHOD.

The apparatus, patterned after the well-known Bragg X-ray spectrometer, needs no special description. The X-ray tube used was of the Coolidge type, the filament of which was operated by a well-insulated storage battery. For the short wave-length work was used a tube of the radiation type, specially exhausted. For the longer wave-lengths a water-cooled tube was employed, thus giving sufficient energy for accurate measurements in the neighborhood of 1.0 Å. Voltage was applied direct to the tube, without rectification, from the secondary of a high tension transformer, the primary of which was connected to a specially controlled 60 cycle A.C. dynamo. Under normal conditions the variations in voltage, over long periods was never greater than 0.1 per cent. The radiation type tube was operated at as high as 110 kv. r.m.s.

The target was approximately 45 cm. from the NaCl crystal. In line with target and crystal, and between the two, were two micrometer slits with lead jaws limiting the X-ray beam to a few tenths of a millimeter in width. The absorbing material to be studied was placed between these two slits, thus avoiding the troublesome secondary radiation when the absorber is placed next to the ionization chamber.

Above .15 Å. the crystal was used in the ordinary way, *i.e.*, the X-ray beam was reflected from the surface planes of a thin crystal. At shorter wave-lengths the glancing angle becomes rather small, of the order of 1 degree, and considerable trouble was experienced from stray secondary radiation entering the chamber directly. It was found that this was in part avoided if a crystal about three millimeters thick were employed,

Richtmyer, PHYS. REV. The writer wishes to express his indebtedness to Dr. Kerr Grant of Adelaide University, South Australia, who collaborated in the early part of this investigation and thereby made many of these measurements possible.

[1] Phil. Mag., 34, p. 270 (1917).
[2] Proc. Roy. Soc., 94, p. 567 (1919).
[3] Proc. Roy. Soc., 94, p. 339 (1919).
[4] Ann. de ph., XIV., p. 49 (1920).

and placed as shown in Fig. 1, thus using the atom planes at right angles to the major surfaces of the crystal. For these wave-lengths the crystal is sufficiently transparent, but cuts off the fluorescent radiation of longer wave-lengths coming from slit edges, crystal mounting, etc.

Fig. 1.

The ionization chamber was of brass, lead covered, approximately 60 cm. long by 10 cm. diameter. Rays were admitted through a thin mica window attached with sealing wax. A micrometer slit, with lead jaws immediately in front of the chamber, protected it from stray radiation. The collecting rod was supported by sulphur insulation, with suitable guard rings. Methyl bromide vapor was found to be most satisfactory for filling the chamber. This substance boils at 4° C. and is rather troublesome to handle but it did not attack the insulation as did some of the other vapors used. The collecting rod was charged to a potential of 200 volts by dry cells.

The ionization current was measured by observing the rate of increase of potential of the insulated system by means of an electroscope of the Bumstead type, the plates of which were charged to + 225 and − 225 volts respectively. The sensitivity was of the order of 200 divisions per volt. The rate of drift of the electroscope leaf was observed by a stop watch.

In order to eliminate observational and instrumental errors as far as possible the following precautions were observed.

1. The intensity of the X-ray beam was so adjusted (by current through filament or by applied voltage) that the rate of drift of the electroscope leaf was of the order of one or two divisions per second, some 60 divisions being covered for each reading. Errors in time measurements were of the order of 0.2 per cent. for each reading. The mean of several readings was taken for each observation. Correction was made for natural leak which was usually less than one per cent. of the ionization current measured.

2. The thickness of the absorber was so chosen that the intensity of the transmitted beam was seldom less than 40 per cent. or more than 65 per cent. of the incident beam.

3. When making an observation at any angle θ, and wave-length λ, the voltage applied to the tube was so adjusted that there was not present in the X-ray beam second order energy of wave-length $\lambda/2$. This was determined by the equation

$$\lambda_{max} = \frac{12.3}{V},$$

where λ_{max} is the wave-length in Ångströms corresponding to the maximum frequency produced by V kilovolts. For example, if one is working at 0.4 Å. there should not be present in the incident beam any radiation of second order of 0.2 Å. Or

$$V = \frac{12.3}{.2} = 61.5 \text{ kilovolts}$$

is the (peak) voltage just necessary to produce radiation of 0.2 Å. Consequently a voltage somewhat less than this is to be employed. To guard against errors in estimating the maximum permissible voltage, due to distortion of wave form, a check was frequently made by observing whether, with the maximum permissible voltage as estimated, there was any radiation at $\theta/2$.[1]

The method of making observations was briefly as follows: At any given (chamber) angle 2θ, the time required for the gold leaf, after breaking earth connection, to drift from division A to division B, say 60 divisions apart, was observed, several readings being taken. The absorbing material was then put in place and several more readings taken. The absorber was removed and the first set repeated. The reciprocals of the mean average time with and without the absorber give quantities proportional to the respective ionization current. The absorption constant, μ, was then determined from the usual law

$$I = I_0 e^{-\mu x}.$$

From knowledge of the density of the absorber and its thickness, or of the mass per square centimeter of absorbing material in the path of the X-ray beam, the mass absorption coefficient μ/ρ was determined.

The following table will make the observations and method of computation clear.

Determination of μ for Molybdenum.

30.3 kilovolts; 1.70 milliamperes.

Absorber: Molybdenum No. 1 + No. 2; ρx = .0358 gms. per sq. cm. Time to charge from division 20 to division 100:

	Absorber out	Absorber in	Absorber out
	32.0 sec.	64.8 sec.	31.8 sec.
	32.2	64.2	31.6
	31.8	64.2	31.6
Mean time		64.4	31.85
1/T (proportional to current)		.0314	.0155
Corrected for leakage		.0311	.0152

$$\frac{\mu}{\rho} = \frac{1}{\rho X} \log_e \frac{I_0}{I} = \frac{2.30 \times .311}{.0358} = 20.0 \text{ gms.}$$

[1] Assuming $\theta = \sin\theta$ in the equation $n\lambda = 2d \sin\theta$.

The center of the direct beam was located on the divided circle by ionization measurements and the wave-length corresponding to any *chamber* angle 2θ was computed from the well known equation

$$\mu\lambda = 2d \sin \theta,$$

where d was taken as 2.814×10^{-8} Å. The calibration was always checked by setting on the K_α lines of tungsten.

TABLE I.

λ	μ/ρ for					
	H_2O	Al	Cu	Mo	Ag	Pb
.09 Å	.165					
.120	.175					
.135	.179	.193	.59	1.51		
.184	.199	.239	1.25	3.27	4.26	3.20
.209	.207	.276	1.74	4.55	5.94	5.11
.234	.220	.328	2.31	6.15	9.50	7.00
.258	.234	.393	3.00	8.10	12.5	9.35
.283	.249	.469	3.87	10.75	15.6	12.0
.308	.266	.567	4.87	13.5	19.5	15.5
.332	.284	.670	6.02	16.7	23.6	19.3
.356	.306	.795	7.29	19.8	28.5	23.7
.368	.319	.861	8.10	21.7	31.0	26.3
.381	.335	.940	8.85	23.7	34.0	29.0
.393	.348	1.017	9.64	25.6	37.0	31.9
.405	.358	1.095	10.2	27.5	39.2	34.5
.417	.376	1.174	11.3	29.5	43.7	37.9
.430	.394	1.285	12.3	31.5	47.8	41.4
.453		1.477	13.4	37.3	56.0	47.0
.479		1.706	16.6	42.5		55.0
.503		1.95	19.0	49.4	11.5	60.8
.551		2.49	24.9	63.8	15.1	73.0
.600		3.18	31.6	80.7	19.6	
.650		4.05	39.5	15.1	24.3	
.715		5.32	52.8	19.9		
.743		5.83		22.6		
.798				27.2		
.847				32.3		
.895				36.0		
.943				43.4		

DATA AND RESULTS.

The following substances have been studied: water, aluminum, copper, molybdenum, silver, and lead, over wave-lengths ranging from .093 Å. to .95 Å. Owing partly to experimental difficulties, however, not all of these substaces have been studied over the entire range. Thus measurements on lead were not made at wave-lengths greater than .550 Å. because of the difficulty of making homogeneous lead foil of the order of .001 in. thick, to say nothing of the difficulty of measuring, to 1 per cent., the thickness after it is made. And in passing, it may be remarked that the securing of homogeneous absorbing films and of measuring their mass per square centimeter, constituted one of the serious difficulties of this investigation.

Table I. and Figs. 2 to 8 show the values of μ/ρ as a function of wave-length for the substances studied. In the table, the data given for aluminum are actual observed points. Because of the fact that measurements were not in general made at the same wave-length settings for all the substances, some of the values given for the substances other than aluminum were read from curves plotted through observed points. Figures 2–8 however are plotted from actual observations, and show μ/ρ plotted as a function of both λ and λ^3. With few exceptions the values of μ/ρ are accurate to one per cent., as may be judged from the smoothness of the curves.

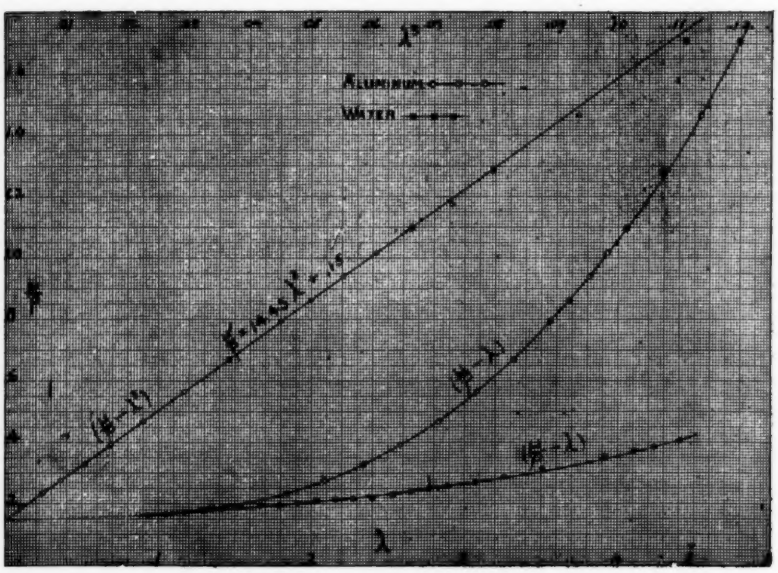

Fig. 2.

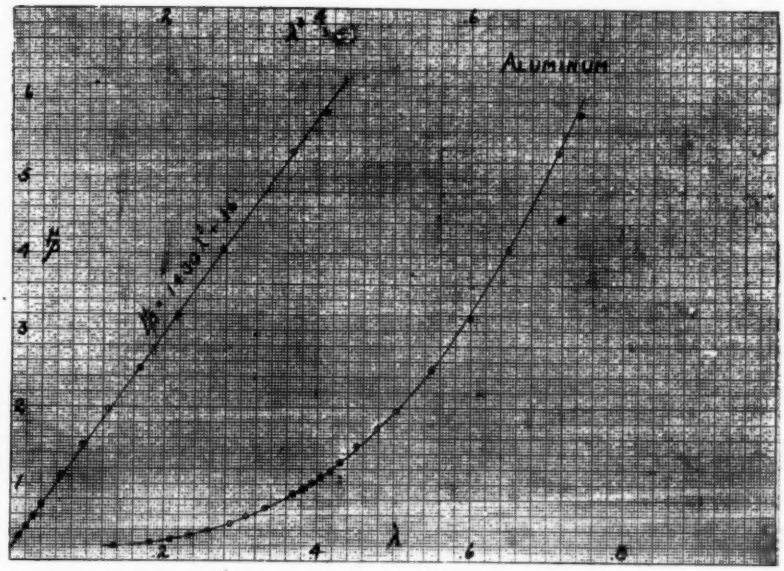

Fig. 3.

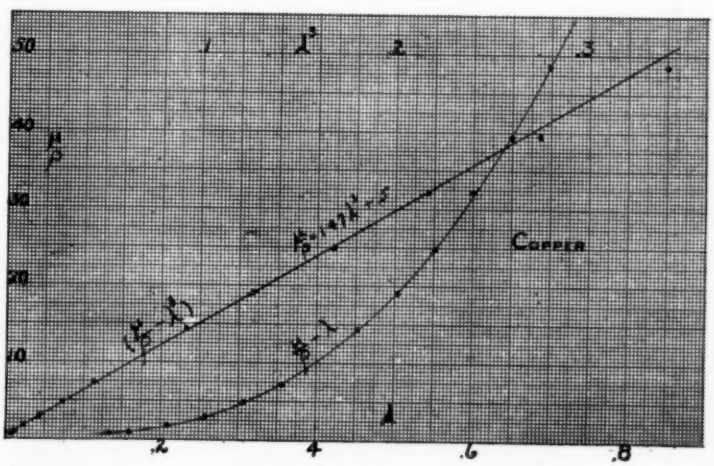

Fig. 4.

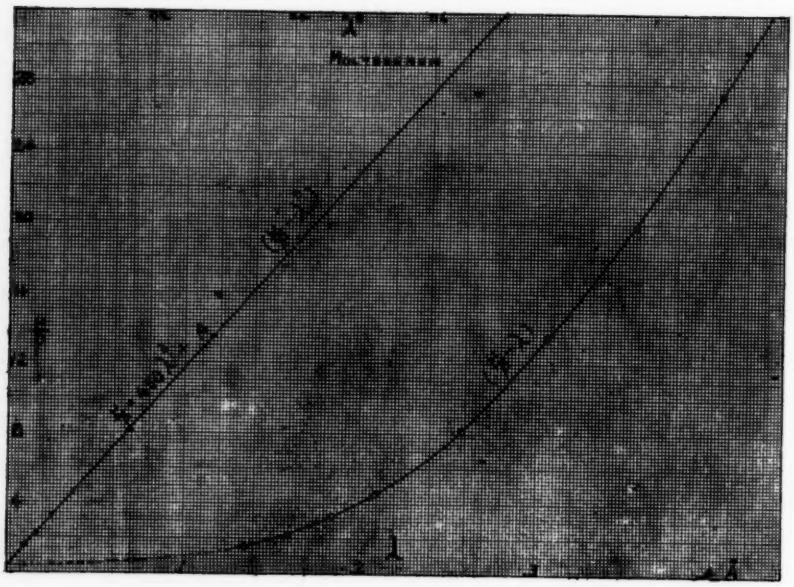

Fig. 5.

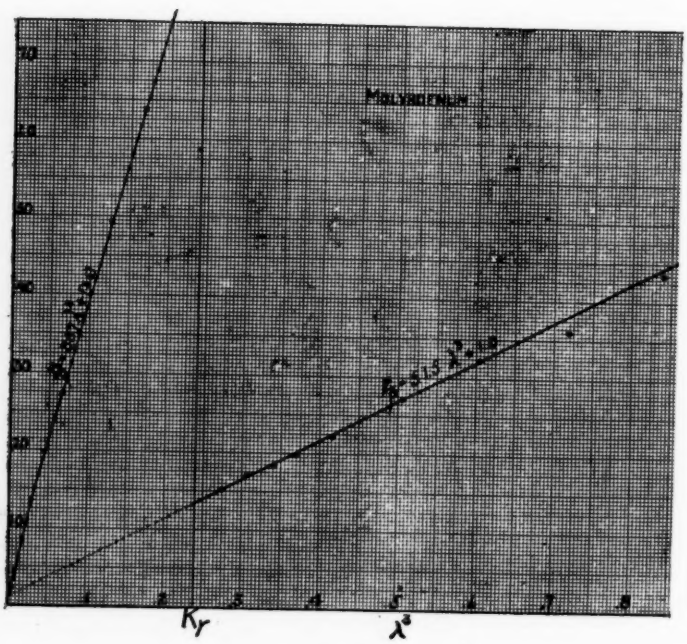

Fig. 6.

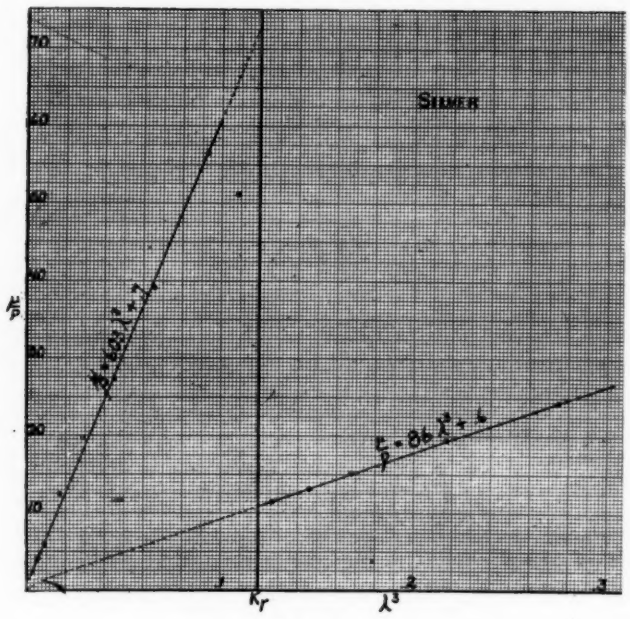
SILVER
$\frac{\mu}{\rho}$
$\frac{\mu}{\rho} = 66.1^{2.7} + 2$
$\frac{\mu}{\rho} = .86\lambda^3 + 4$
K_r
λ^3

Fig. 7.

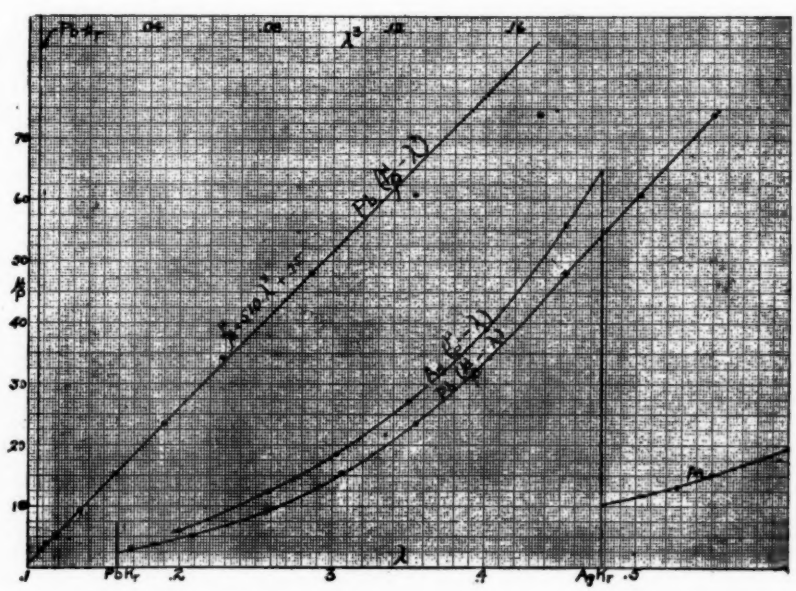
$Pb\,K_r$
λ^3
$\frac{\mu}{\rho}$
$\frac{\mu}{\rho} = 210.1^{2.75}$
$Pb\,\frac{\mu}{\rho} = 46.0$
$Ag\,\frac{\mu}{\rho} = 11$
$Pb\,\frac{\mu}{\rho} = 23$
Ag
λ
$Pb\,K_r$
$Ag\,K_r$

Fig. 8.

CONCLUSIONS.

I. *Variation of μ/ρ with Wave-length and with Atomic Number of Absorber.*—It will be seen in general that the curves plotted with μ/ρ as a function of λ^3 are, at the shorter wave-lengths, straight lines; but that as one goes toward longer wave-lengths, the values of μ/ρ as predicted by this straight line are increasingly too great. Thus in Fig. 2 the equation

$$\left(\frac{\mu}{\rho}\right)_{Al} = 14.45\lambda^3 + .15$$

represents μ/ρ very accurately up to about .4 Å., but above that value of λ values of μ/ρ are less than would be expected. In Fig. 3, which is an extension of μ/ρ for aluminum to longer wave-lengths the same tendency is noticed. The equation

$$\left(\frac{\mu}{\rho}\right)_{Al} = 14.30\lambda^3 + .16$$

gives values of μ/ρ slightly too high at .4 Å. and slightly too low at .7 Å.

Reference to copper, Fig. 4, shows the same tendency to concavity toward the λ^3 axis. This concavity is still more marked in the case of substances of higher atomic number, as is shown by Figs. 5 and 7 for molybdenum and silver respectively.

Figures 6, 7 and 8 show the values of μ/ρ for molybdenum, silver, and lead respectively on the long wave-length side of the K absorption line. Here again the linear relation between μ/ρ and λ^3 seems to hold rather accurately, except that in the case of lead the last two observations fall very much below the straight line.

It may be concluded therefore that the general equation

$$\frac{\mu}{\rho} = F\lambda^3 + \frac{\sigma}{\rho} \qquad (1)$$

represents the law of absorption, provided one is not approaching, from the short wave-length side, a characteristic absorption band. To express the law accurately there must be deducted from the right hand side of (1) a term involving some unknown function of λ/λ_K where λ_K is the wave-length corresponding to the K absorption limit. Keeping in mind this limitation the following equations give the mass-absorption coefficient for the several substances studied, and the range over which the equation holds.

In order to ascertain how μ/ρ varies with the atomic number of the absorber it is necessary to discuss separately the two terms $F\lambda^3$ and σ/ρ. Since it is the atom as a whole which acts as a unit in absorption it is

TABLE II.

On Short Wave-length Side of the Respective K Absorption Limit.

Aluminum.....	$\mu/\rho = 14.45\lambda^3 + .15$	.1Å to .4Å
	$\mu/\rho = 14.30\lambda^3 + .16$	.4 to .7
Copper.....	$\mu/\rho = 147\lambda^3 + .5$	.1 to .6
Molybdenum ..	$\mu/\rho = 450\lambda^3 + .4$	.1 to 35
Silver[15].....	$\mu/\rho = 603\lambda^3 + .7$	.1 to .4

On Long Wave-length Side of the K Absorption Limit.

Molybdenum ..	$\mu/\rho = 51.5\lambda^3 + 1$	
Silver[1]	$\mu/\rho = 86\lambda^3 + .6$	
Lead.....	$\mu/\rho = 510\lambda^3 + .75$	

λ is expressed in Ångström units.

first necessary to reduce values of μ/ρ, in which absorption is given for a layer of material thick enough to contain 1 gram per sq. cm., to the atomic absorption coefficient, μ_a, for a layer containing one atom per sq. cm. This is done by dividing both sides of equation (1) by n/A, the number of atoms per gram, where n is Avogadro's constant 6.06×10^{23}, and A is the atomic weight. This gives

$$\mu_a = \frac{\mu/\rho}{n/A} = \frac{A}{n} \cdot F\lambda^3 + \frac{A}{n} \cdot \frac{\sigma}{\rho}, \tag{3}$$

$$= F_a\lambda^3 + \sigma_a \tag{4}$$

where $F_a = F(A/n)$, and $\sigma_a[= (\sigma/\rho) \cdot (A/n)]$ is the atomic scattering coefficient.

Figure 9, plotted from the last two columns of Table III. shows the relation between F_a and the atomic number of the absorber. The slope

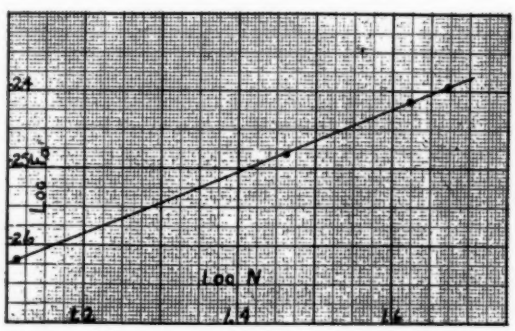

Fig. 9.

[1] Data on Silver is approximate, since measurements were not made as carefully as on other substances.

TABLE III.

Substance.	At. Number.	F.	$F_a = F\,A/n$.	Log N.	Log F_a.
Al.....	13	14.45	64.5×10^{-23}	1.114	−27.810
Cu.....	29	147	1,550	1.462	−25.190
Mo.....	42	450	7,130	1.623	−25.854
Ag.....	47	650[1]	11,600	1.672	−24.053

of the curve is 4.00. We may accordingly write

$$F_a = kN^4. \tag{5}$$

The mean value of k for these four elements is 2.29×10^{-27}. Putting this value of F_a in equation (4) gives

$$\mu_a = 2.29 \times 10^{-27} N^4 \lambda^3 + \sigma_a \tag{6}$$

as the value of the atomic absorption coefficient as a function of atomic number and wave-length, *provided* the values of λ are well removed (*i.e.*, on the short wave-length side) from the K characteristic absorption limit. Although of numerical significance only, it may be observed that this value of k is about one third of Planck's constant h. So that for keeping the equation in mind we may write, approximately

$$\mu_a = \frac{h}{3} N^4 \lambda^3 + \sigma_a. \tag{6'}$$

Equation (5) agrees with the conclusions of Owen,[5] who, however, worked at only one wave-length.

It must be kept in mind, as pointed out above, that (6) does not represent the value of μ_a as the absorption limit is approached. To make the equation complete we should have to write

$$\mu_a = kN^4\lambda^3 + \sigma_a - f\left(\frac{\lambda}{\lambda_K}\right), \tag{7}$$

where $f(\lambda/\lambda_K)$ is a function, at present unknown, but which increases, perhaps rapidly, as the K absorption limit λ_K is approached.

An inspection of Table II. and the curves for molybdenum, silver, and lead shows that on the long wave-length side of their characteristic K absorption limits an equation of the form

$$\frac{\mu}{\rho} = F'\lambda^3 + \frac{\sigma}{\rho} \tag{8}$$

expresses the relation between μ/ρ and λ. Data is not sufficiently accurate (on account of the very thin absorbing sheets necessary) to deter-

[1] Taken from the short wave-length data for Ag.

mine how F_a' depends on atomic number. But a plot, similar to that for F_a shows that

$$F_a' = k'N^{4.4}.$$

All that can be said is that the exponent of N is at least of the same order as that in the equation for F_a.

II. *The Value of the Mass Scattering Coefficient.*—From Fig. 2 it is seen that the value of the mass scattering coefficient of aluminum, σ/ρ, is .15. Data for the remaining elements is not sufficiently accurate to determine values of σ/ρ except to say that there seems to be a tendency for an increase in σ/ρ with increasing atomic number of absorber. Much more data at very short wave-lengths is necessary to determine how σ/ρ varies with N.

But it is to be noted, that, so far as data goes, the value of σ/ρ as determined from data on silver and molybdenum is to a first approximation the same on the long wave-length side of the K absorption limit as on the short wave-length side, for both curves cut the σ/ρ axis at approximately the same point.

The importance of this is shown in Fig. 10, in which the value of μ/ρ is shown diagrammatically as a function of λ^3. Allow X-rays of wave-length λ_1 to fall on a *thin* sheet of the absorber whose characteristic K limit is λ_K. The vertical line *ad* is proportional to the total energy removed from the transmitted beam. Of this energy an amount proportional to *ab* is re-radiated as K fluorescent radiation; an amount proportional to *bc*, as L fluorescent radiation; and an amount proportional to *cd* is scattered without change of wave-length. Subject therefore to the correction term mentioned in equation (7) the mass absorption coefficient for a given substance is given by

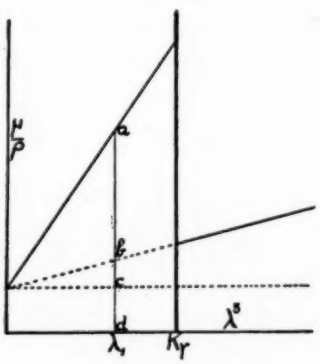

Fig. 10.

$$\frac{\mu}{\rho} = K\lambda^3\Big]_0^K + L\lambda^3\Big]_0^L + \frac{\sigma}{\rho}. \qquad (9)$$

The term $K\lambda^3]_0^K$ is to be used from $\lambda = 0$ to the K absorption limit and determines the K fluorescent radiation; the term $L\lambda^3]_0^L$ is to be used up to the first L absorption limit. L itself is probably made up of several terms.

In spite of the correction term mentioned in equation (7) these laws of X-ray absorption are exceedingly simple as compared with the corre-

sponding laws in the ultra-violet and visible region of the spectrum. It may not be too much to hope that a gradual extension of these laws to longer and longer wave-lengths may lead to some clue as to absorption of visible radiation.

III. *Is there Evidence of a Discontinuity in Absorption Curves Indicative of a Characteristic "J" Radiation?*—Table IV. gives the location of the discontinuities in the curves for μ/ρ as observed by various investigators.

TABLE IV.

N	Element.	Barkla and White.[10]	Williams.[11]	Owen.[12]	Dauvillier.[13]
6.....	C	.42 Å.	—	—	—
7.....	O	.39	—	—	—
13.....	Al	.37	.49 Å.	—	.358
14.....	Si	—	—	.493	—
29.....	Cu	—	.448	—	—
35.....	Br	—	—	—	.227
47.....	Ag	—	—	—	.178

Barkla and White's data is shown in Fig. 11.[17] They assume that the absorption of water is due mainly to oxygen; and of paraffin mainly to carbon. In the present paper no data is shown for carbon, but reference to Fig. 2 will show that there is no discontinuity in water at .39 Å.

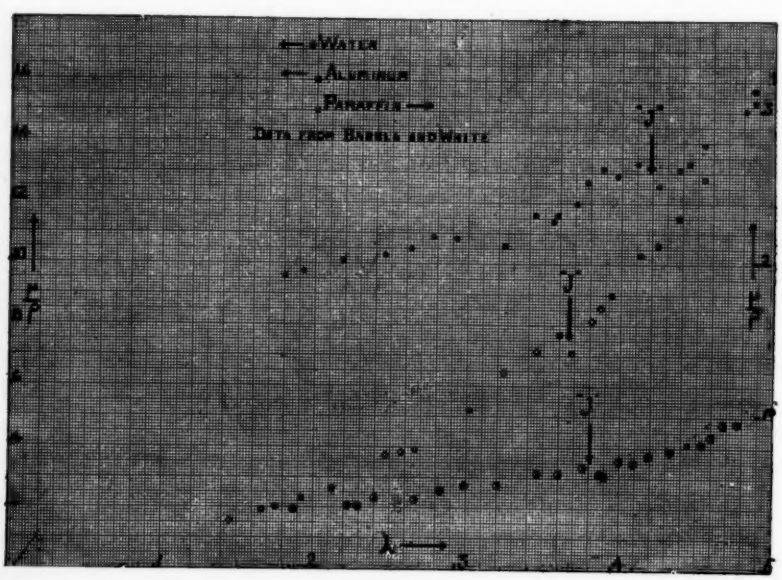

Fig. 11.

[17] The plotting is my own. F.K.R.

Williams concluded that Barkla's discontinuity for aluminum was at too short a wave-length, and should be at .49 Å. Fig. 3 shows no discontinuity in aluminum at this wave-length, nor is there any apparent discontinuity in copper at .448 Å. as will be seen from Fig. 4. It is suggested that these apparent discontinuities were due to a rather large experimental error, combined with an accidental grouping of points. Thus Barkla and White's data would show apparent discontinuities at several other wave-lengths than those mentioned.

Dauvillier's apparent discontinuity in aluminum at .358 seems to be due to the *K* absorption limit of iodine in his ionization chamber, for no discontinuity is to be observed at this point in Fig. 2. His "*J*" for bromine is identical with Duane's α_3 line for tungsten. I have been unable to trace the origin of his suggestion of a "*J*" for silver at .178 Å. But Dauvillier states that if the square root of the frequencies for these three "*J*'s" be plotted, after Mosley, against atomic number, a straight line results. Such a graph predicts a discontinuity for copper at .250 Å. and for molybdenum at .196 Å. An inspection of Figs. 4 and 5 will show that there are no discontinuities for these elements at these points. The conclusion is that there is no evidence in favor of discontinuities in absorption curves indicating a radiation in the X-ray region shorter than the *K* series.

The Mass Absorption Coefficient of Water at Very Short Wave-lengths.

It is known that the value of μ/ρ for the hard rays from radium *C* is of the order of .05 to .08 for various substances, much lower than would be expected by extrapolation to $\lambda = 0$ of the curves, say, in Fig. 2 for water and aluminum. The wave-length of these rays from radium *C* is not known but has been estimated by Rutherford and Andrade to be less than .01 Å. An attempt was made to see whether, in the case of water, there was any evidence of a downward bend in the curve, at very short wave-lengths, toward the value given for the rays from radium *C*. Although measurements were pushed down to .093 Å. there was no evidence of a downward bend. It is as if the mechanism operative at 0.1 Å. must differ materially from that which is responsible for the scattering of the much shorter rays from radium *C*. If we assume that the innermost group of electrons around the nucleus is responsible for the emission and absorption of *K* radiation, we must look to the nucleus itself for the emission and absorption (scattering) of still shorter groups of rays as the hard rays from radium *C* are thought to be. Rutherford has recently found evidence pointing to a structure of the nucleus, of elements

up to oxygen, as made up of both positive and negative charges. Such a system should obviously both radiate and absorb. Perhaps further knowledge of the nucleus may throw light on the question of absorption of very short waves.

SUMMARY.

1. It has been shown that the mass absorption coefficient of Ag, Al, Cu, Mo, Pb and H_2O for X-rays is given in each case by an equation of the form

$$\frac{\mu}{\rho} = F\lambda^3 + \frac{\sigma}{\rho}, \tag{1}$$

so long as the values of λ are considerably smaller than the K absorption limit λ_K, and that as this limit is approached a term of the form $f(\lambda/\lambda_K)$ must be subtracted. This function is unknown.

2. Above the K absorption limit an equation of the same type holds, namely

$$\frac{\mu}{\rho} = F'\lambda^3 + \frac{\sigma}{\rho}. \tag{8}$$

3. If the atomic absorption coefficient μ_a be computed, equation (1) becomes

$$\mu_a = F_a\lambda^3 + \sigma_a \tag{4}$$

and it has been shown that F_a is proportional to the fourth power of the atomic number, *i.e.*,

$$F_a = kN^4,$$

where the numerical value of k is 2.29×10^{-27}. This combines with (1) to give the atomic absorption coefficient of substances (from Al to Ag at least) by the equation

$$\mu_a = 2.29 \times 10^{-27} N^4\lambda^3 + \sigma_a.$$

4. The value of the mass scattering coefficient seems to be independent of wave-length, since to a first approximation at least the value of σ/ρ is the same, for each substance, in equations (8) and (1).

5. There is no evidence, on the basis of absorption measurements, for the existence of a "J" radiation.

6. The value of μ/ρ for water, down as far as .093 Å. shows no evidence of an approach to the corresponding value for the hard rays from radium C.

CORNELL UNIVERSITY,
ITHACA, N. Y.

THE EFFECT OF A UNIFORM MAGNETIC FIELD ON THE MOTION OF ELECTRONS BETWEEN COAXIAL CYLINDERS.

By Albert W. Hull.

Synopsis.

Equations of motion for electrons are developed starting from a cylindrical cathode and moving toward a co-axial cylindrical anode, in a uniform magnetic field parallel to the common axis. The electrons will reach the anode if the ratio of potential difference to magnetic field is greater than a critical value, and will fail to reach it if the ratio is less than this value.

In the case of a *small cathode* in the axis of an anode of radius R at potential V, the critical magnetic field is

$$H = \sqrt{\frac{8m}{e}}\frac{V^{1/2}}{R}.$$

For a *small anode* of radius r_0 at potential V, in the axis of a cathode of radius R_0, the critical field is

$$H = \frac{r_0}{R_0}\sqrt{\frac{8m}{e}}\frac{V^{1/2}}{R_0} + \frac{2mv_0}{eR_0}.$$

In this case the initial velocity v_0 of the electrons cannot be neglected. This equation also applies, with appropriate R_0 and e/m, to *positive ions* produced by electrons from an *internal cathode*.

If the radii of *both cylinders are large* the solution reduces to the familiar one of plane parallel plates.

The *equation of the path* of the electrons is deduced, on the assumption that the space charge distribution is the same as without magnetic field. The path is given by $r = R(\sin\frac{3}{2}\theta)^{2/3}$. This is a close approximation to the true path, as calculated from the space charge distribution recently worked out by Langmuir. Experimental curves showing current at constant potential as a function of magnetic field, for different anode diameters, voltages, filament temperatures, degrees of symmetry, etc., are in agreement with theory within the limit of experimental error.

The *internal cathode* tube offers a means of measuring e/m, and the *internal anode* a means of measuring the distribution of initial velocities.

Introduction.

THE motion of electrons in uniform electric and magnetic fields has been studied by many investigators.[1] A few special cases of non-uniform fields have also been investigated; for example, the magnetic field due to current in a long straight wire,[2] and the electric field between a straight filament and concentric cylinder. In the latter case

[1] See, for example, J. J. Thomson, Conduction of Electricity through Gases, second Ed., pp. 104–116.

[2] J. J. Thomson, l.c., p. 108; O. W. Richardson, Roy. Soc. Proc., 90, 174, 1914.

the effect of the space charge of the moving electrons has been taken account of.[3]

A case of special interest, which appears not to have been investigated, is that in which the electric field is radial and symmetrical about a straight line, and the magnetic field is uniform and parallel to this line. This applies to an evacuated tube containing a straight filament and a concentric circular cylinder, in the axis of a long straight solenoid. The theoretical conditions can be accurately realized, and the apparatus is easy to construct and operate.

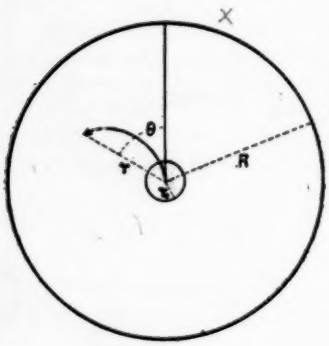

Fig. 1.
Small cathode in axis of cylindrical anode.

Relation between Magnetic Field, Voltage, and "Range" of the Electrons.—The equations of motion of an electron starting from a point at distance r_0 from an axis (Fig. 1), with initial velocity components

$$u_0 = \left(\frac{dr}{dt}\right)_0; \quad v_0 = r_0\left(\frac{d\theta}{dt}\right)_0; \quad w_0 = \left(\frac{dz}{dt}\right)_0;$$

and moving under the influence of a constant magnetic field of intensity H parallel to the axis, and a symmetrical radial electric field of intensity F (function of r only) are:

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = F\frac{e}{m} - H\frac{e}{m}r\frac{d\theta}{dt}, \tag{1}$$

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = H\frac{e}{m}\frac{dr}{dt}, \tag{2}$$

$$\frac{d^2z}{dt^2} = 0. \tag{3}$$

Integrating (2) with respect to time gives:

$$\frac{d\theta}{dt} = \frac{r_0^2}{r^2}\left(\frac{v_0}{r_0} - \frac{He}{2m}\right) + \frac{He}{2m}. \tag{4}$$

The maximum radial distance from the axis that an electron can travel may be found either by equating the total work eV done by the electric force to the kinetic energy of tangential motion

$$\frac{1}{2}m\left(r\frac{d\theta}{dt}\right)^2,$$

given by Eq. 4; or by equating the radial velocity dr/dt to zero. The

[3] Langmuir, PHYS. REV., 2, 450, 1913; Schottky, Ann. d. Phys., 44, 1011, 1914.

latter is found from integrating Eq. 1 with respect to r, which gives:

$$\left(\frac{dr}{dt}\right)^2 = 2\frac{e}{m}V_r - H^2\left(\frac{e}{2m}\right)^2 r^2\left(1-\frac{r_0^2}{r^2}\right)^2 - H\frac{e}{m}r_0v_0\left(1-\frac{r_0^2}{r^2}\right) + v_0^2\left(1-\frac{r_0^2}{r^2}\right) + u_0^2, \quad (5)$$

where

$$V_r = \int_{r_0}^{r} F dr$$

is the potential difference between the starting point r_0 and the point r.

Putting $dr/dt = 0$ gives the maximum distance r_m from the axis that the electron can reach, in terms of the potential V_r at this distance and the magnetic field H:

$$V_r = H^2\frac{e}{8m}r_m^2\left(1-\frac{r_0^2}{r_m^2}\right)^2 + \left(\frac{Hr_0v_0}{2} - \frac{v_0^2}{2e/m}\right)\left(1-\frac{r_0^2}{r_m^2}\right) - \frac{u_0^2}{2e/m}. \quad (6)$$

If a conducting cylinder of radius r_m is inserted in the position of the surface r_m, all the electrons will reach it if its potential is greater than V_r, and none will reach it if its potential is less than V_r. *Hence if one observes the current to the cylinder as a function of V with constant H, or as a function of H with constant V, this current should fall abruptly to zero at the critical values of V and H.* This is found to be true experimentally (Figs. 7–10).

The above equations refer to the motion of both electrons and ions, in either direction, starting at any point with any initial velocity. They become much simpler under practical conditions. Four special cases will be considered.

CASE 1. INTERNAL CATHODE.—*Electrons start from a straight filament of radius r_0 and travel to a concentric cylindrical anode of radius R, large compared with the filament (Fig. 1).*

Equation 6 then becomes

$$V = H^2\frac{e}{8m}R^2 + H\frac{r_0v_0}{2} - \frac{u_0^2 + v_0^2}{2e/m}. \quad (7)$$

The last term on the right-hand side represents the initial energy of emission of the electrons, expressed in the same units as V.

If the initial energies are those of thermal emission, between zero and 2 volts, then the last two terms are negligible for the great majority of electrons, whose initial energies are less than 1/10 volt; and they are negligible even for electrons with the highest initial energies when V and H are large.

For example, in the case of a 4 mil filament in the axis of a 1 inch cylinder,

$$r_0 = .005 \text{ cm.}; \qquad R = 1.25 \text{ cm.};$$
$$\sqrt{u_0^2 + v_0^2} = 2 \times 10^7 \text{ (0.1 volt initial energy).}$$

If $H = 10$,

$$\left.\begin{aligned} V &= 3.5 \times 10^8 \pm 5 \times 10^5 - 10^7 \text{ E.M.U.} \\ &= 3.5 \pm .005 - 0.1 \text{ volts} \\ \text{If } H &= 100 \\ V &= 350 + .05 - 0.1 \text{ volts,} \end{aligned}\right\} \text{ initial energy } \frac{1}{10} \text{ volt,}$$

while for the very fastest electrons, with initial energies of 2 volts ($\sqrt{u_0^2 + v_0^2} = 8 \times 10^7$).

$$\left.\begin{aligned} &\text{If } H = 10, \; V = 3.5 \pm .02 - 2 \text{ volts} \\ &\text{If } H = 100, \; V = 350 \pm 0.2 - 2 \text{ ``} \\ &\text{If } H = 1000, \; V = 35000 \pm 2 - 2 \text{ ``} \end{aligned}\right\} \text{ initial energy 2 volts.}$$

It is seen that the term involving the first power of H is negligible under all conditions, so that Eq. 7 becomes

$$V = H^2 \frac{e}{8m} R^2 - \frac{1}{2} \frac{m}{e} (u_0^2 + v_0^2). \tag{7'}$$

The only correction due to initial velocities is the addition of the initial energy of emission, $\frac{1}{2}m(u_0^2 + v_0^2)$ to the work eV done by the electric field, as in all thermionic applications. This correction is negligible at reasonable voltages. Hence in all practical cases, for electrons moving from a straight filament to a concentric cylinder,

$$V = H^2 \frac{e}{8m} R^2 \tag{8}$$

or

$$H = \sqrt{\frac{8m}{e}} \frac{V^{1/2}}{R}. \tag{9}$$

V = potential difference between filament and cylinder.

R = radius of cylindrical anode.

H = magnetic field just sufficient to prevent electrons reaching anode.

The value of H corresponding to the middle of the steep portion of the $I - H$ curve (Figs. 7–10) can be accurately measured, so that Eq. 9 may be used for the experimental determination of e/m.

If e/m is taken as 1.77×10^7 E.M.U. and V is expressed in volts, H and R in Gauss and cm. resp., Eqs. (8) and (9) become

$$V = .0221 \, H^2 R^2, \tag{8'}$$

$$H = 6.72 \frac{V^{1/2}}{R}. \tag{9'}$$

CASE 2. *Internal Anode.—Electrons start from an outer cylinder of radius R_0 and move toward a concentric inner cylinder or wire of radius r_0, small compared with R_0 (Fig. 2).*

Eq. 6 becomes in this case:

$$V = \beta^2 \frac{e}{8m}\left(HR_0 - 2\frac{m}{e}v_0\right)^2 \quad (10)$$

or

$$H = \frac{1}{\beta R_0}\sqrt{\frac{8m}{e}}\,V^{1/2} + \frac{2mv_0}{eR_0}. \quad (11)$$

V = potential difference between outer and inner cylinders, reckoned positive when the inner cylinder is positive.

R_0 = radius of outer cylinder.

H = magnetic field just sufficient to prevent electrons reaching inner cylinder.

β = ratio of radii of outer and inner cylinders.

v_0 = tangential initial velocity of electrons.

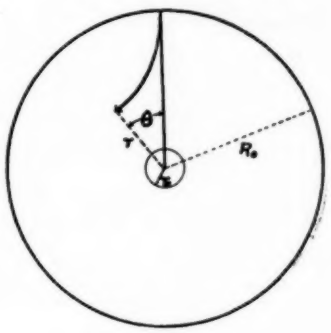

Fig. 2.

Small anode in axis of cylindrical cathode.

For electrons with very small tangential initial velocities this becomes:

$$V = \beta^2 \frac{e}{8m} H^2 R_0^2. \quad (12)$$

This is the same as Eq. 8 for electrons starting from the filament, except for the factor β^2. The voltage against which the electrons can travel in the same magnetic field is β^2 times greater than in case 1; or, conversely, with the same potential difference between cathode and anode, the magnetic field required to prevent electrons reaching the anode is only $1/\beta$ as great with internal anode as with internal cathode. This is found to be true experimentally (see Figs. 15–20).

The effect of initial velocities is much greater in this case, however, than in the case of internal cathode, so that in practical cases only a very small fraction of the electrons have a sufficiently small initial velocity to obey Eq. 12. For the rest the critical magnetic field will depend on the initial velocity, and the curves representing current as a function of magnetic field will not be steep, as in the case of electrons starting from the filament, but will fall off slowly (see Figs. 15–20).

The magnitude of the correction for initial velocities can best be illustrated by an example. If the diameters of inner and outer cylinders are in the ratio 1 : 25, viz., 12 mils (r_0 = .0152 cm.) and 3/10 inches

(R = 0.38 cm.) respectively, and the potential difference 50 volts, Eqs. 7 and 11 give, for the cases of external anode and internal anode respectively,

$$H_1 = \frac{\sqrt{8m/e}}{R}\sqrt{V+\frac{1}{2}\frac{m}{e}(u_0^2+v_0^2)}$$

$$= 17.7\sqrt{50+2.82\times 10^{-16}(u_0^2+v_0^2)} \quad \text{(external anode)},$$

$$H_2 = \frac{\sqrt{8m/e}}{\beta R}V^{1/2}+\frac{2m}{eR}v_0 = 5+3\times 10^{-7}\,v_0 \quad \text{(internal anode.)}$$

The following table gives the values of H_1 and H_2 for different values of v_0 (taking $u_0 = 0$).

v_0		H_1 (External Anode).	H_2 (Internal Anode).
Cm./sec.	Volts.	Gauss.	Gauss.
0×10^7.....	0.	125	5.0
1 "	.028	125 + .035	5.0 ± 3.0
2 "	.112	125 + .141	5.0 ± 6
3 "	.25	125 + .317	5.0 ± 9
4 "	.45	125 + .563	5.0 ± 12
5 "	.70	125 + .880	5.0 ± 15
6 "	1.01	125 + 1.27	5.0 + 18
7 "	1.37	125 + 1.72	5.0 ± 21
8 "	1.80	125 + 2.25	5.0 ± 24

r_0 = .0152 cm.; R = 0.38 cm.; β = 25, V = 50 volts.

CASE 3. *Parallel Planes.—The radii of both cylinders are large compared with their distance apart.*

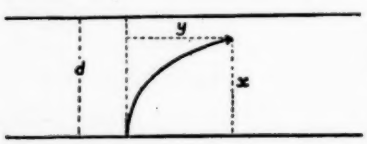

Fig. 3.
Plane parallel plates.

This applies to electrons moving between plane parallel plates, and includes the case of uniform electric and magnetic fields at right angles to each other.

Putting

$$r - r_0 = x; \qquad rd\theta = dy; \qquad \frac{r}{r_0} = 1;$$

Eq. 6 becomes

$$V_x = H^2\frac{e}{2m}x^2 + Hv_0x - \frac{u_0^2}{2e/m}, \tag{13}$$

where x is the maximum distance that an electron starting from the plane $x = 0$, with initial velocity u_0 and v_0 in the x and y directions respectively, can travel from this plane in a magnetic field H parallel to Z; and V is the potential at the point x.

With large values of H and x the effect of initial velocities is small, and (13) becomes

$$V = H^2 \frac{e}{2m} d^2 , \tag{14}$$

where d is the distance between plates, V the potential difference between them, and H the critical magnetic field that is just sufficient to prevent the electrons reaching the anode plate.

This value of V is 4 times greater than that against which electrons starting from a filament are able to travel the same distance in the same magnetic field (Eq. 8); or, for electrodes the same distance apart and at the same potential difference, only half as strong a magnetic field is required to turn electrons back in the case of parallel plates as is the case of filament and concentric cylindrical anode.

The effect of initial velocities is much greater in the case of the parallel plates, however, than for the filament. For example, for plates 1 cm. apart and electrons with 2 volts initial energy ($v_0 = 8 \times 10^7$), Eq. (13) gives

$$\left.\begin{array}{l} \text{for } H = \ 10, \ V = \ 8.8 \pm \ 8 - 2 \\ \text{“ } H = 100, \ V = 880 \pm 80 - 2 \end{array}\right\} \text{ 2 volt electrons}$$

while even for electrons of only 1/10 volt initial energy ($v_0 = 2 \times 10^7$)

$$\left.\begin{array}{l} \text{for } H = \ 10, \ V = \ 8.8 \pm \ 2 - 0.1 \text{ volts} \\ \text{“ } H = 100, \ V = 880 \pm 20 - 0.1 \ \text{“} \end{array}\right\} \frac{1}{10} \text{ volt electrons.}$$

Hence, the curve representing current as a function of magnetic field will not be as steep in the case of plane parallel plates as in the case of filament and cylindrical anode.

CASE 4. *Positive ions are produced at a point r cm. from a straight filament where the potential is V with respect to the filament, with initial velocities u_0 and v_0, and move toward the filament in a magnetic field H.*

Equations (10) and (11) apply, with

$$\frac{e}{m} = \frac{10^4}{M} ,$$

where M is the molecular weight of the ion. The small value of e/m, as compared with that for electrons, more than offsets the factor β^2, so that a magnetic field which is capable of turning back electrons moving from a filament to concentric cylindrical anode will not be able to prevent the positive ions produced by these electrons from reaching the filament, except in the case of small diameter filament (large β) and ions of low atomic weight. The relative effect, upon the value of critical magnetic field, of initial velocities and voltage respectively is the same as in the

case of electrons, for the same initial energy per unit charge. Hence the cutoff by a magnetic field of positive ions moving toward a filament will not be sharp except in the case of very strong magnetic and electric fields, and will be practically ineffective in the case of small diameter filaments and heavy ions.

PATHS OF THE ELECTRONS.

The phenomena discussed thus far are independent of the potential distribution in the tube. The complete determination of the paths of the electrons requires a knowledge of this potential distribution, which depends on the space charge of the electrons. Certain properties of the motion can be obtained, however, without this knowledge, and an approximate solution can be obtained on the assumption that the space charge distribution is the same as that which exists without the magnetic field. There is experimental evidence that this assumption is not far from the truth.

The angular velocity of the electron is given by Eq. 4 above;

$$\frac{d\theta}{dt} = \frac{r_0^2}{r^2}\left(\frac{v_0}{r_0} - \frac{He}{2m}\right) + \frac{He}{2m}. \tag{4}$$

In the case of an electron starting from a filament of radius r_0 with initial velocity

$$v_0 \leq r_0 \frac{He}{2m},$$

the first term on the right hand side becomes negligible compared with the second as soon as the electron has travelled a small distance r equal to about 10 times the radius of the filament. From this point on its angular velocity is constant and equal to $He/2m$. In the usual case this distance is of the order of 1/2 mm. For example, in the case of a 1 mil fil. (r_0 = .0012 cm.) and electrons with initial velocities of 2×10^6 cm./sec., if H = 200 gauss,

$$\frac{v_0}{r_0} = 1.7 \times 10^9; \qquad \frac{He}{2m} = 1.8 \times 10^9; \qquad 10r_0 = .012;$$

the electron acquires 99 per cent. of its maximum angular velocity within 1/10 mm. of the filament. Hence

$$\frac{d\theta}{dt} = \frac{He}{2m} \left\{ \begin{array}{l} r > .01 \text{ cm.}, \\ H > \dfrac{2mv_0}{er_0}, \end{array} \right. \tag{15}$$

that is, for electrons starting with small initial velocity from a 1 mil filament, *except in the very small region within 1/10 mm. of the filament, that angular velocity of the electrons is constant and equal to He/2m.*

$v\left(\frac{cm.}{sec.}\right) = 1.88 \times 10^{-24}\sqrt{X}$; where X = # of e.m.

If an electron starts with initial angular velocity v_0/r_0 large compared with $He/2m$, this angular velocity will fall off rapidly at first, at a rate proportional to $1/r^3$, then less rapidly as it approaches the limiting value $He/2m$, and will not reach its limiting value in as short a distance as that given by Eq. (15). For example, in the case of the 1 mil filament and magnetic field of 200 gauss considered above, electrons with 1/10 volt initial energy ($v_0 = 2 \times 10^7$ cm./sec.) must travel 1 mm. before their angular velocity falls to within 1 per cent. of its constant value, and electrons with 2 volts initial energy ($v_0 = 8 \times 10^7$) must travel 4 mm. before their angular velocity is constant within 1 per cent.

From Eqs. 4 and 5 one obtains for the angle ϕ between the radius vector and the tangent to the path, neglecting initial velocities:

$$\tan \varphi = \frac{r d\theta}{dr} = \pm \frac{H \frac{e}{2m} (r^2 - r_0^2)}{\sqrt{2 \frac{e}{m} V r^2 - \left(H \frac{e}{2m} \right)^2 (r^2 - r_0^2)}}. \qquad (16)$$

Since V is a function of r only, this has the same numerical value for an electron traveling in toward the axis as for one traveling out, hence *the electrons return in paths of exactly the same shape as those in which they go out.* This shows that the electrons, if they fail to reach the anode, do not continue circling around the cathode in approximately circular orbits forever (or until they strike gas molecules), but return periodically to the cathode. The total angle they describe before returning depends on the distribution of electric intensity as a function of r. If the space charge distribution were the same as without magnetic field this angle would be 270 degrees. The actual space charge, which has recently been calculated by Langmuir,[1] is slightly greater than without magnetic field and gives a total angle of 288.5 degrees.

There are two special cases of intensity distribution that lead to very simple solutions of the path. The first represents the initial condition, before space charge has had time to build up. The second represents very nearly the stationary condition.

CASE 1. *Space charge is negligible.*

The electric intensity is that due to the potential difference V between the cylinders, viz.,

$$F_r = \frac{V}{r \log \frac{R}{r_0}},$$

where R and r_0 are the radii of cylinder and filament respectively. If r_0

[1] These calculations will be published soon by Dr. Langmuir.

is very small, the intensity is negligible except very close to the filament. The electron will therefore move with constant velocity, and its path will be a circle of radius

$$\frac{\sqrt{2\frac{m}{e}V}}{H}.$$

This solution represents the initial condition, before the electrons which fail to reach the cylinder have had time to pile up in the space in appreciable quantity. The steady state, however small the emission, must be that of full space charge.

CASE 2. *Space Charge the Same as Without Magnetic Field.*—Consider the case of electrons starting with zero initial velocities from a filament of very small diameter. The potential at any distance r from the axis is given by the equation[1]

$$i_0 = \frac{2\sqrt{2}}{9}\sqrt{\frac{e}{m}}\frac{V^{3/2}}{r}, \tag{17}$$

where i_0 is the maximum electron current per cm. length from the filament to a concentric cylinder of radius r at potential V. Equations (4) and (5) give

$$\left.\begin{aligned}\left(\frac{d\theta}{dt}\right)^2 &= \frac{\alpha^2}{r^4}(r^2 - r_0^2),\\ \left(\frac{dr}{dt}\right)^2 &= 2\frac{e}{m}V - \frac{\alpha^2}{r^2}(r^2 - r_0^2)\end{aligned}\right\}; \alpha = H\frac{e}{2m},$$

from which, if $r_0 = 0$,[2]

$$\frac{d\theta}{dr} = \frac{\alpha}{\sqrt{2e/m\ V - \alpha^2 r^2}}$$

and substituting the value of V from (17)

$$\frac{d\theta}{dr} = \frac{\alpha}{\sqrt{(9i_0 e/m)^{2/3}r^{2/3} - \alpha^2 r^2}}. \tag{18}$$

Eq. (18) may be written,

$$d\theta = \frac{3/2dx}{\sqrt{\beta^2/\alpha^2 - x^2}}, \text{ where } \begin{cases} x = r^{2/3}, \\ \beta^2 = \left(9i_0\frac{e}{m}\right)^{2/3}, \\ \alpha = \frac{He}{2m}. \end{cases}$$

[1] Langmuir, PHYS. REV., 2, 455, 1913.

[2] The assumption that $r_0 = 0$ leads to no mathematical discontinuities. The solution thus obtained (Eq. 19) will therefore represent the actual solution, viz., the solution for finite r_0, to any desired degree of accuracy, if r_0 is taken sufficiently small.

The integral of this is

$$\theta = \frac{3}{2}\sin^{-1} x\,\frac{\alpha}{\beta} + \theta_0 .$$

Substituting the values of α, β and x, and determining the constant θ_0 from the condition that θ shall be zero when r is zero,[1]

$$\theta = \frac{3}{2}\sin^{-1}\left(\sqrt[3]{\frac{e^2}{72 i_0 m^2}}\,H r^{2/3}\right)$$

or

$$r = 6\sqrt{2 i_0}\,\frac{m}{e}\,H^{-3/2}\left(\sin\frac{\theta}{3}\right)^{3/2}, \qquad (19)$$

substituting for i_0 its value from Eq. 17 in terms of the diameter R and potential V_0 respectively of the outer cylinder,

$$r = 4\sqrt[4]{2}\left(\frac{m}{e}\right)^{3/4}\frac{V^{3/4}}{R^{1/2}H^{3/2}}\left(\sin\frac{\theta}{3}\right)^{3/2}. \qquad (20)$$

If r_{max} represents the maximum value of r in Eq. 20, that is the greatest radial distance the electron can travel in magnetic field H when the potential on the anode, of radius R, is V_0, Eq. (20) may be written

$$r = r_{max}\left(\sin\frac{\theta}{3}\right)^{3/2}, \qquad (21)$$

where

$$r_{max} = 4\sqrt[4]{2}\left(\frac{m}{e}\right)^{3/4}\frac{V^{3/2}}{R^{1/2}H^{3/2}}.$$

When $r_{max} = R$, *i.e.*, when the electrons can just reach the outer cylinder, this reduces to

$$R = \sqrt{\frac{8m}{e}}\,\frac{V^{1/2}}{H},$$

which is identical with Eq. (9).

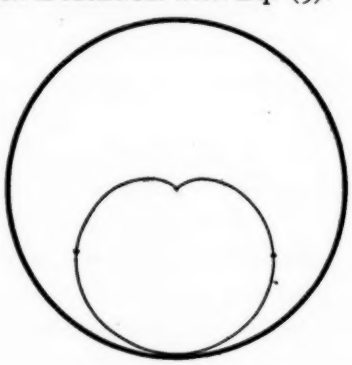

Fig. 4.

Plot of approximate equation of path, $r = R[\sin^{2/3}\theta^{3/2}]$.

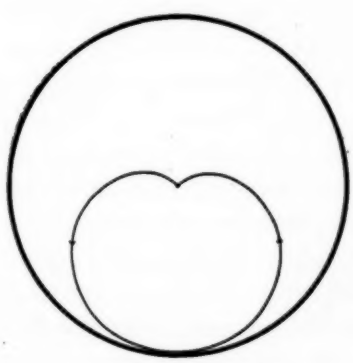

Fig. 5.

Exact path of electron (Langmuir's solution).

[1] See footnote 2, previous page.

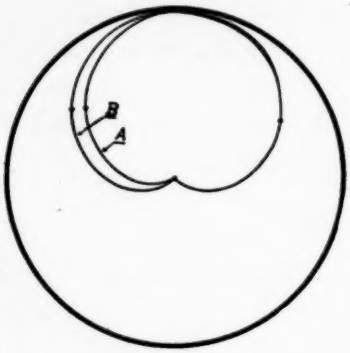

Fig. 6.

Path of electron. Degree of approximation represented by Eq. 21.

A, approximate path, Eq. 21; *B*, exact path (Langmuir).

The path given by Eq. 20 is shown in Fig. 4. The exact path, as calculated by Langmuir, is shown in Fig. 5. For the purpose of comparison the two are plotted together in Fig. 6. The difference is inappreciable up to $r = 9/10\ r_{max'}$, *i.e.*, Eq. 21 represents the true path very closely except in the region near the maximum range. In this region the radial velocity is less than that given by Eq. 21, making the total angle traversed per cycle greater, viz., 288.5 degrees instead of 270 degrees.

EXPERIMENTS.

The experimental work was done in coöperation with Mr. E. F. Hennelly. More than 1,000 tubes have been studied, of various shapes and sizes, all conforming as closely as possible to the symmetry conditions postulated above. The following examples are typical:

A. Internal Cathode. Straight Filament in Axis of Cylinder. General Characteristics.—The diameter of the filament is always small compared with that of the anode, so that r_0^2/R^2 is negligible compared with unity. Hence, the relation between critical magnetic field, voltage, and anode radius is that given by equation (9):

$$H = \sqrt{\frac{8m}{e}}\,\frac{V^{1/2}}{R}\,. \qquad (9)$$

H is the value of the magnetic field that is just sufficient to prevent electrons from reaching an anode of radius R at potential V with respect to filament. If V is kept constant and H gradually increased, the current should remain constant until H reaches the critical value given by equation (9), and should then fall abruptly to zero.

Figure 7, for which I am indebted to Dr. K. H. Kingdon, shows a typical example. The tube contained a straight tungsten filament .012 cm. in diameter, about 1 mm. from the axis of an anode 3.86 cm. in diameter, 12 cm. long. A rotating commutator applied voltage and heating current alternately, so that measurements were made while the filament was an equipotential surface. The potential difference between cathode and anode was 50 volts, and the filament temperature high enough so that the current was limited only by space charge.

It is seen that the current maintains nearly its full space charge value

until the critical magnetic field is reached, and then falls very abruptly. The steepness is as great as to be expected from the degree of symmetry, *i.e.*, with filament 1 mm. out of center. The effect of high initial velocities is observable near the bottom of the curve.

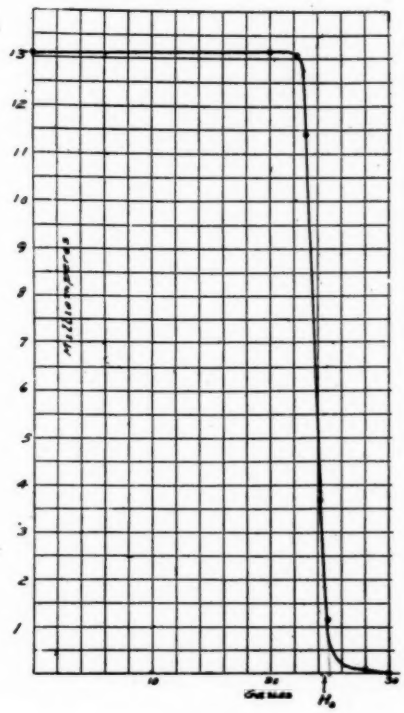

Fig. 7.

Typical Gauss-Ampere characteristic at 50 volts.

The theoretical "cut off" field is shown by the dotted curve, as calculated from Eq. 11, viz.,

$$H = 6.72\sqrt{\frac{50}{1.93}} = 24.56 \text{ Gauss.}$$

Effect of Heating Current through Filament.—The characteristic of Fig. 7 was taken with a rotating commutator, adjusted to cut off the heating current during the brief intervals when measurements were being made. If measurements are made while the heating current is flowing, the curves are less steep, due to the voltage drop along the filament. This effect is more marked at low than at high anode voltages. Most of the data which follows was taken with filament current flowing.

No effect due to the magnetic field of the heating current has been observed.

Effect of Anode Voltage.—Fig. 8 shows the characteristic of a tube with anode 3/4 in. diameter × 1½ in. long and cathode 4 mils diameter, at three different anode potentials, viz., 10.2, 21.7, and 57 volts. These measurements were taken by Dr. K. H. Kingdon with rotating commutators.

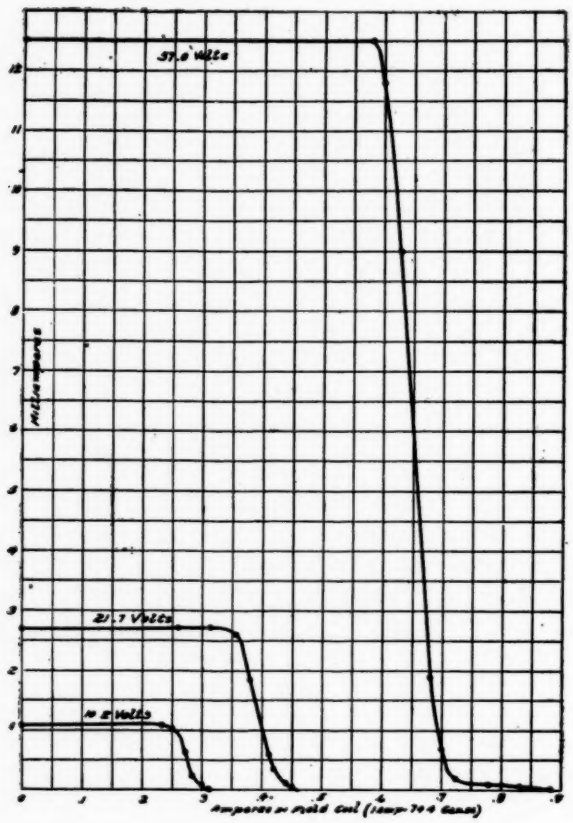

Fig. 8.

Characteristic of ¾ inch dia. anode, at different potentials.

Fig. 9 shows the $I - H$ characteristic of a tube with 1 in. × 4 in. anode and 10 mil cathode at four different anode voltages, 60, 125, 200, and 250 volts. The potential drop in the filament was 10 volts, and anode potentials were measured from the negative end of the filament. Hence in the 60-volt test, one end of the filament was at 50 volts with respect to the anode, and the other at 60.

It will be observed that the maximum currents in Figs. 8 and 9 are in the ratio of the 3/2 power of the voltage, and the critical magnetic fields are in the ratio of the 1/2 power of the voltage.

Figs. 10 and 11 show the lower portions of curves taken at higher potentials. The anode of Fig. 10 was 2 in. diam. × 2 in. long, that of Fig. 11 was 1 in. diam. × 4 in. long. The upper portions of these curves could not be obtained because of the excessive heating of the anode. The full current at 9,000 volts would have been 100 amps. Its reduction to 35 milliamperes was due solely to the magnetic field.

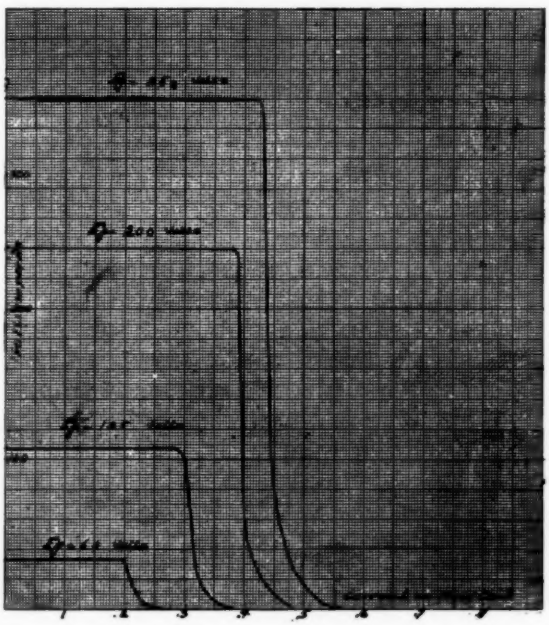

Fig. 9.

Characteristic of 1 inch dia. anode, at different potentials.

Effect of Filament Temperature.—Fig. 12 shows the characteristics of a tube of the same construction as that of Fig. 9, at three different filament temperatures. The potential difference was 250 volts. The steep portions of the curves are identical. There is a slight transition region between the horizontal and vertical portion of each curve. In the upper curve the current is limited by space charge, in the other two by filament temperature.

Effect of Diameter of Anode.—Fig. 13 shows the characteristics, at 250 volts, of 4 tubes of different diameters, from 2 inches to 3/8 inch. In order to bring them within the limits of the same plot, the currents in the smaller tubes have been limited by filament temperature to the same maximum value as for the largest diameter. The critical magnetic fields are proportional to the diameters within the limits of accuracy with which these diameters were known.

Effect of Symmetry.—Symmetry, both of electric and magnetic field, is essential for sharp cut-off. A numerical estimate of the effect of asymmetry is difficult, but an example will illustrate it. Fig. 14 shows two characteristics taken with the same tube. In curve *A* the filament was parallel to the magnetic field; in curve *B* the magnetic field was inclined 20 degrees to the filament.

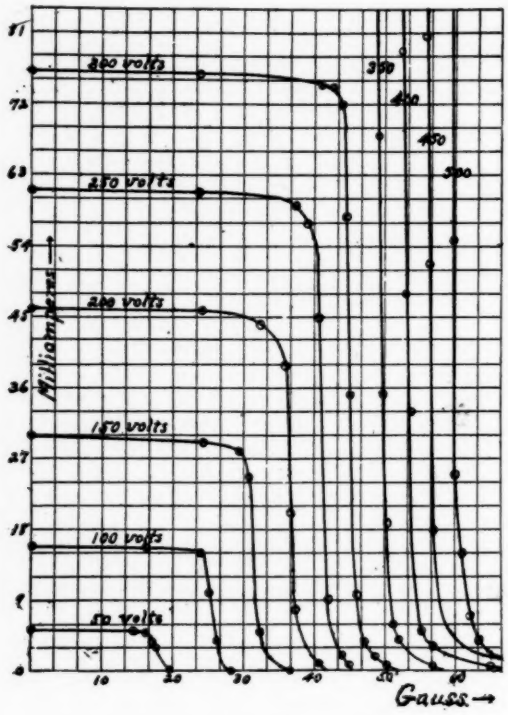

Fig. 10.

Characteristic of 2 inch dia. anode, at different potentials.

The effect of small pieces of iron in the magnetic field is very striking. With the magnetic field adjusted to the critical value, the point of a pen brought near the tube is sufficient to reduce the current to half value.

Effect of Gas.—The equations for cut-off are independent of potential distribution in the tube. The cut-off should not be affected, therefore, by the presence of positive ions, even in sufficient numbers to completely neutralize the space charge of the electrons, provided the mean free path is large compared with the dimensions of the tube. Hence with low gas pressure one should observe currents which are not limited by space charge, but are completely controllable by a magnetic field. At higher gas pressures the magnetic field becomes less and less effective, due to

the fact that every electron which collides with a molecule will suffer a change, in direction at least, of its velocity, and a large fraction of these velocity changes will be in such direction as to require a stronger field to keep the electrons from the anode. It is probable, also, that the impact of positive ions on the cathode gives rise to the emission of electrons of relatively high initial velocity, some of which, according to Eq. 7, require a stronger field to turn them back.

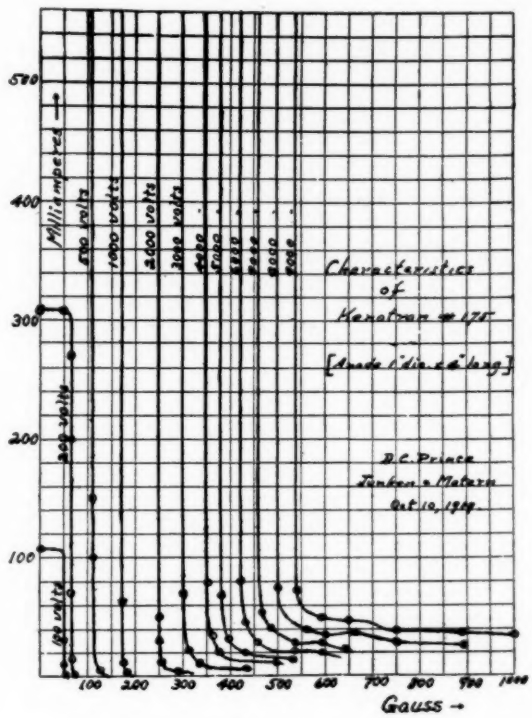

Fig. 11.

Characteristic of 1 inch dia. anode, at high potentials.

Figure 15 shows the typical behavior of tubes containing small amounts of gas. Curve *A* is the Gauss-ampere characteristic in good vacuum. The maximum current is limited by space charge, which increases slightly, with corresponding decrease in current, as the critical magnetic field is approached. The steepness of cut-off is limited mainly by symmetry and potential drop along the filament. Curve *B* shows the effect of a few hundredths of a bar of gas. The space charge of electrons is partially neutralized by positive ions so that the maximum current is slightly larger, especially in the neighborhood of critical magnetic field, where the paths of the electrons are longer and the ionization greater. The cut-off

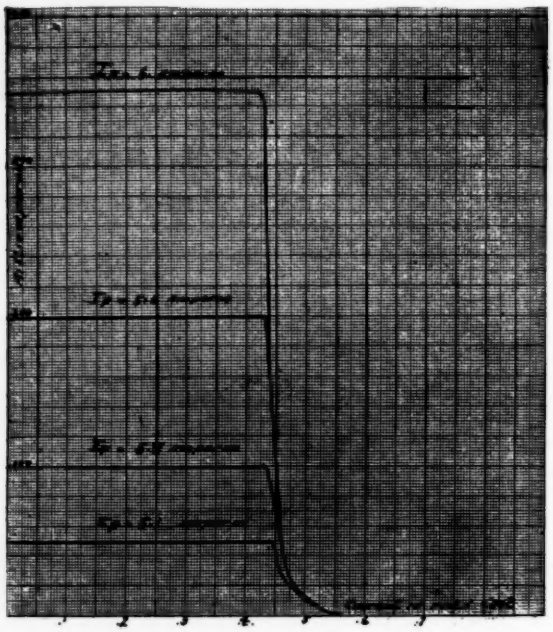

Fig. 12.

Characteristic of 1 inch dia. anode, at 250 volts, at different filament temperatures.

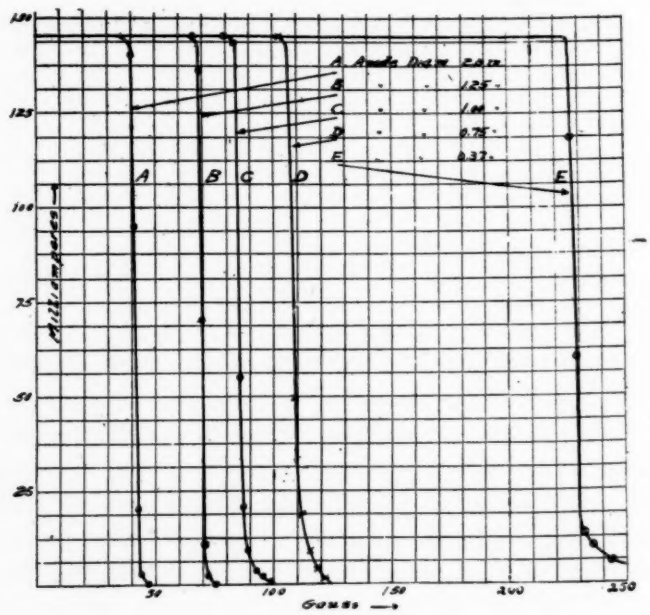

Fig. 13.

Characteristics of different anodes, at 250 volts.

is not appreciably affected. Curve *C* shows the effect of a larger amount of gas, of the order of 1 bar. The maximum current is very large, but the critical magnetic field is only slightly greater than for good vacuum. The curve has a long tail, however, representing the electrons which have collided or have abnormal initial velocities, and hence require a field much stronger than the critical value to keep them from the anode. At higher pressures, of the order of 10–20 bars, the magnetic cut-off is much less steep (Curve *D*) and a considerable fraction of the electrons are entirely uncontrollable by the magnetic field. At still higher pressures magnetic fields of this order of magnitude are practically ineffective.

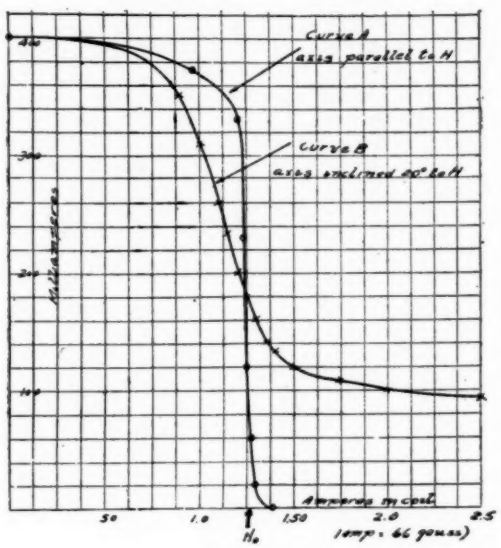

Fig. 14.

Effect of lack of symmetry. Magnetic field not parallel to axis.

B. Internal Anode. Straight Rod in Axis of Helical Filament. General Characteristics.—Equations (10) and (11) apply. The effect of initial velocities of the electrons is large, except in strong magnetic fields. Hence the curves representing current to the central anode as a function of magnetic field, at constant anode potential, will not be as steep as in the first case. On the other hand, the magnetic field required to produce a marked change in current is very much smaller than in the first case, so that the earth's magnetic field produces an easily measurable effect, and the field of the current which heats the filament practically prevents any electrons reaching the anode at potentials less than about 100 volts.

A typical example is shown in Fig. 16.[1] The cathode was a helical

[1] For this and the following curves I am indebted to Mr. C. G. Found.

coil 0.63 cm. in diameter, 1.7 cm. long, consisting of 10 turns of 10 mil tungsten wire. The anode was a 10 mil wire in the axis of the helix. A rotating commutator applied heating current and voltage alternately, so that the measurements were taken when no current was flowing through the filament. The measurements were taken with the tube vertical, and the earth's magnetic field was not compensated. The abscissas are the values of the impressed magnetic field. It will be noted that the curve is symmetrical, not about the axis of ordinates, but about the ordinate $H = 0.6$ approximately, the vertical component of the earth's magnetic field.

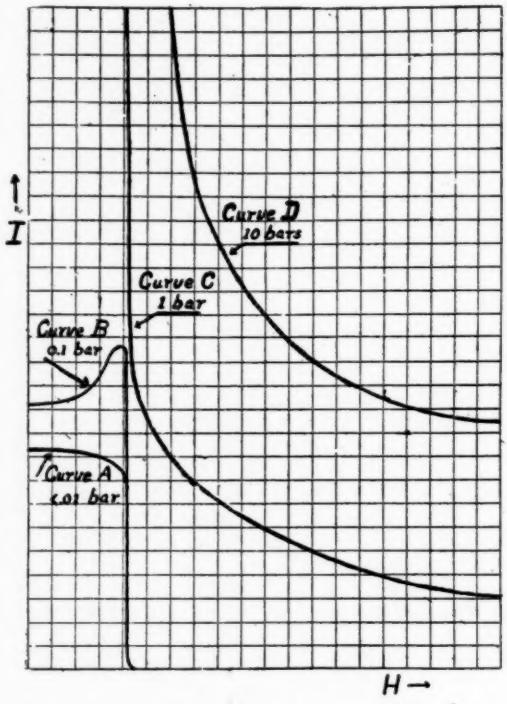

Fig. 15.

Effect of gas pressure upon magnetic cutoff.

The value of magnetic field just sufficient to prevent electrons of zero tangential initial velocity from reaching the anode, according to Eq. 12, is shown at H_0. It is seen that about half of the electrons, viz., those whose initial velocities are in the same direction as the magnetic deflection, are kept from the anode by a field less than H_0. The remainder, whose initial velocities are in the opposite direction to the deflection produced by the magnetic field, require a field greater than H_0.

It is possible to calculate the exact current to be expected at each value of magnetic field, assuming the tangential initial velocities to be distributed according to Maxwell's law.[1] The magnetic field just sufficient to keep back electrons with initial velocities less than v_0' or greater than v_0'' is given by Equation (11), viz.,

$$H = \pm \frac{1}{\beta R_0} \sqrt{\frac{8m}{e} V} + 2 \frac{m v_0}{e R_0} \qquad (11)$$

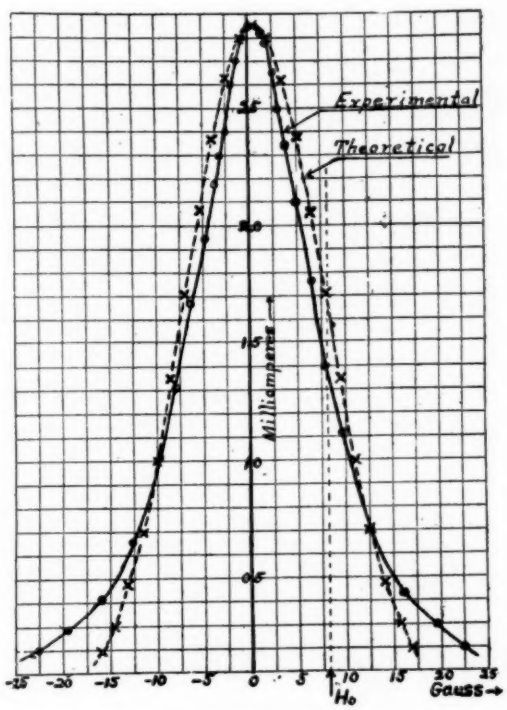

Fig. 16.

Typical internal anode characteristic at 110 volts. O, experimental points (solid curve). ×, theoretical points (broken curve).

v_0' and v_0'' correspond to the plus and minus sign of the radical respectively. In the present case, with $\beta = 25$, $R_0 = .38$ cm. and $V = 110$ volts, this becomes

$$H = \pm 7.5 + 3v_0.$$

The values of H required for different values of v_0 are given in Table II. All electrons with initial velocities *between* any pair of values v_0' and v_0''

[1] Richardson and Cooke, Phil. Mag., 16, 353, 1908.

will *not* be deflected from the anode by the corresponding H, *i.e.*, will reach the anode. This number, which represents the relative current to the anode for each value of H, is given by Maxwell's distribution law as

$$i_H = \int_{v_0'}^{v_0''} f(v_0)dv_0 = \int_{v_0'}^{v_0''} \left(\frac{m}{2\pi R_1 T}\right)^{1/2} e^{-\frac{mv_0^2}{2R_1T}} dv_0 , \qquad (22)$$

where m = mass of electron = 9.01×10^{-28},
R_1 = gas constant for 1 molecule = 1.372×10^{-16} ergs/degree,
T = absolute temperature = 2,200 degrees,
v_0 = initial (tangential) velocity of electron in cm./sec.

The values of i_H given by equation (22) are tabulated in column 4 of Table II. These values are percentages of the total number of electrons emitted by the filament. It is seen that only 83 per cent. of the electrons emitted reach the anode *even without magnetic field*. For comparison with the observed currents the values in column 4, multiplied by 3.44 to reduce them to the same scale, are tabulated in column 5, and plotted as crosses in Fig. 16. The agreement is remarkably good considering the degree of symmetry of the apparatus. With more perfectly constructed apparatus it should be possible to test accurately the law of distribution of initial velocities of the emitted electrons.

TABLE II.

Initial Velocity of Electrons.		Critical Magnetic Field (from Eq. 11).	Fraction of Emitted Electrons with Velocities between v_0' and v_0''.	Number of Electrons Reaching Anode.	
v_0' cm./sec. $\times 10^7$.	v_0'' cm./sec. $\times 10^7$.	H Gauss.	$i_H = \int_{v_0'}^{v_0''} f(v_0)dv_0$.	Milliamperes Calculated $3.44\, i_H$)	Observed Milliamperes.
− 2.5.....	+ 2.5	0	0.828	2.85	2.85
− 2.0.....	+ 3.0	1.5	.812	2.80	2.78
− 1.5.....	+ 3.5	3.0	.765	2.63	2.45
− 1.0.....	+ 4.0	4.5	.692	2.38	2.10
− 0.5.....	+ 4.5	6.0	.600	2.07	1.80
0	+ 5.0	7.5	.497	1.71	1.45
+ 0.5.....	+ 5.5	9.0	.391	1.35	1.15
+ 1.0.....	+ 6.0	10.5	.292	1.01	0.97
+ 1.5.....	+ 6.5	12.0	.207	.71	.72
+ 2.0.....	+ 7.0	13.5	.138	.47	.56
+ 2.5.....	+ 7.5	15.0	.086	.30	.47
+ 3.0.....	+ 8.0	16.5	.050	.17	.39
+ 4.0.....	+ 9.0	19.5	.015	.05	.28

Magnetic Field of Heating Current.—Fig. 17 shows the effect of the

magnetic field of the heating current in the filament. The measurements were taken with a tube having the same cathode as that of Fig. 16, viz. a helix of .63 cm. diam. but with a 40 mil (1 mm.) anode. Curve *A* is taken with rotating commutator, that is the measurements were made while no heating current was flowing. Curve *B* was taken with D.C. heating current of 6.1 amperes.

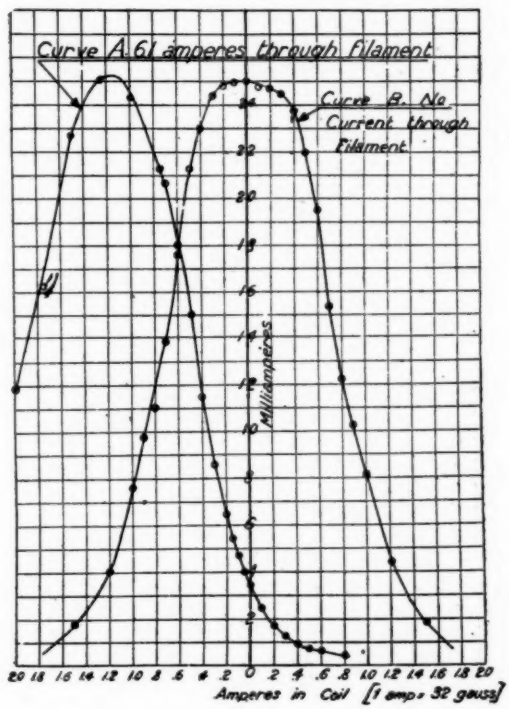

Fig. 17.

Internal anode. Effect of magnetic field of heating current in filament.

The magnetic field of the heating current reduced the electron current to 13 per cent. of its value with no magnetic field. The two curves are similar except that one is displaced to the left by 48 gauss, which is the approximate value of the field produced by the heating current of 6.1 amperes through a coil of 6 turns per cm. (10 turns in 1.7 cm.).

Effect of Voltage.—Figs. 18 and 19 show the variation of current with magnetic field for the same tube at different anode potentials. The tube is the same as of Fig. 17, viz., cathode .63 cm. diameter, 1.7 cm. long, anode 1 mm. diameter.

In Fig. 18 the currents are all plotted to the same scale, showing the relative magnitudes, and the relative variations in milliamperes with

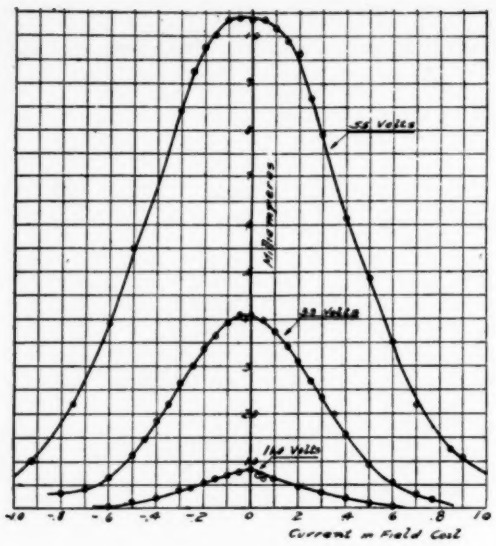

Fig. 18.

Characteristic of 1 mm. internal anode at different potentials.

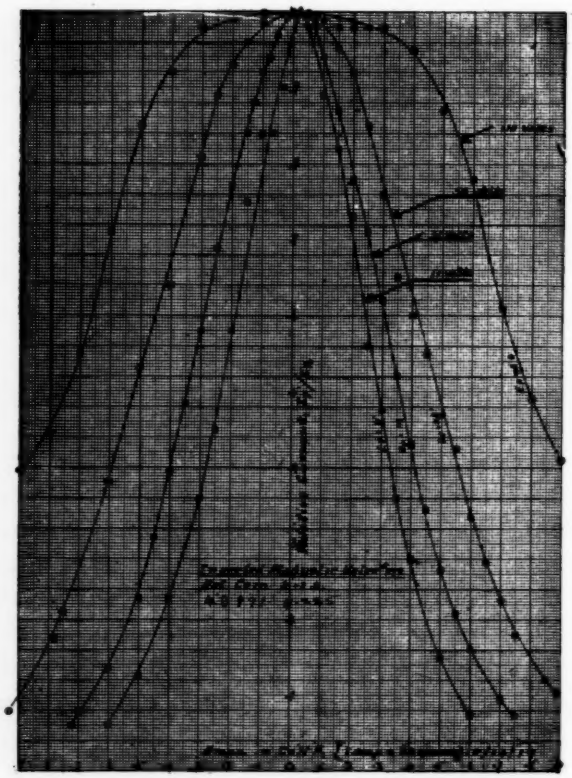

Fig. 19.

Characteristic of 1 mm. internal anode at different potentials. Currents expressed as fractions of maximum at each voltage.

magnetic field. The maximum currents at the different voltages are proportional to the 3/2 power of the voltages.

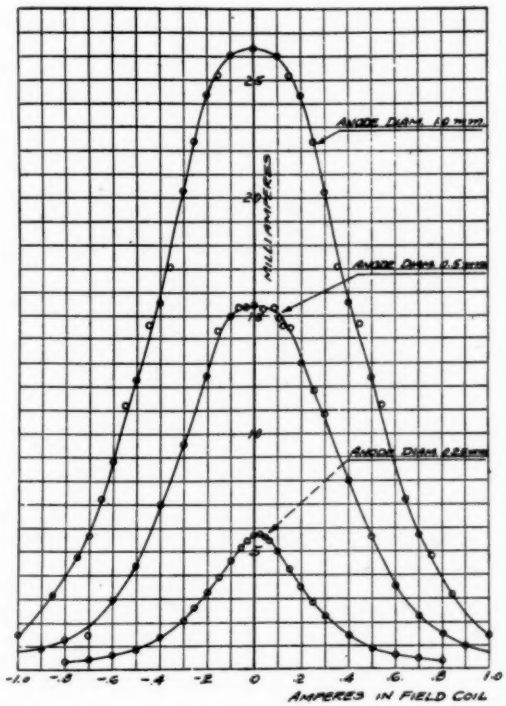

Fig. 20.

Different internal anodes at same potential (110 volts).

In Fig. 19 the maximum at each voltage is taken as unity, in order to show the relative *percentage* variation with magnetic field. It is seen that the field required to produce the same percentage decrease is larger the higher the voltage. At lower voltages, the variation is less rapid than the square root of the voltage, since the initial velocity term in equation 11 predominates. At the highest voltage the curve has a fairly flat top, showing smaller relative effect of initial velocities, and approaching the steep cutoff of internal cathode tubes.

Effect of Anode Diameter.—Figures 20, 21, and 22 show the characteristics of three similar tubes with different anode diameters. The cathodes of the three tubes were the same, viz., 10 turns of 10 mil tungsten wire wound in a helix 0.63 cm. in diam. 1.7 cm. long. The anodes were 0.25 mm., 0.50 mm., and 1.0 mm. diam. respectively.

Figure 20 shows the actual currents in the three tubes at the same anode voltage, plotted to the same scale. The difference in the maxima

is due partly to space charge, but mostly to initial velocities, since by Eq. (10) only part of the electrons emitted can reach the anode even in the absence of magnetic field, viz., those whose initial energy of tangential motion, expressed in equivalent volts, is less than V/β. (V is the potential difference between anode and cathode, and β the ratio of cathode diameter to anode diameter.) Thus, the fraction that can reach the anode falls off very rapidly with increasing β, that is, with smaller anode diameter.

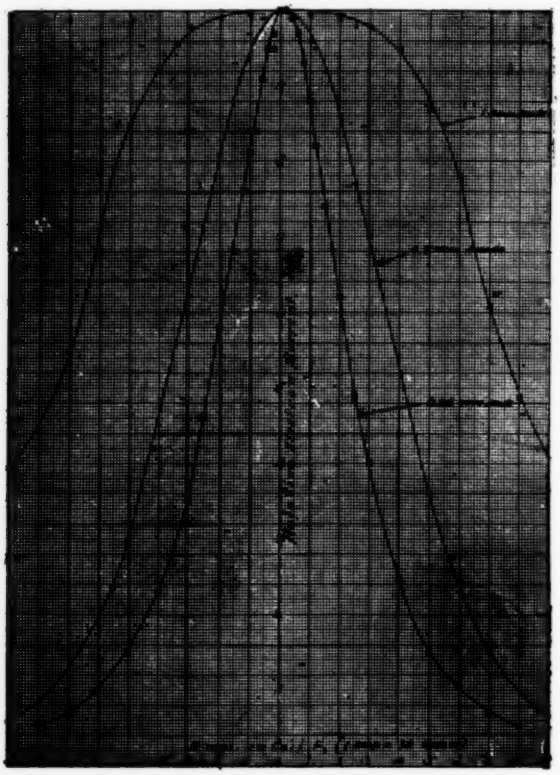

Fig. 21.

Different internal anodes at same potential (110 volts). Currents expressed as fractions of maximum for each anode.

In Fig. 21, the same curves are reproduced on different scales, the maximum of each being taken as unity. It will be noted that the smaller anode diameter gives the greater percentage decrease of current with magnetic field, and that weaker fields are required for smaller anode diameter, as predicted by Eq. 11.

Fig. 22 shows the characteristic of the same three tubes at different voltages, so chosen that the maximum space-charge limited current was

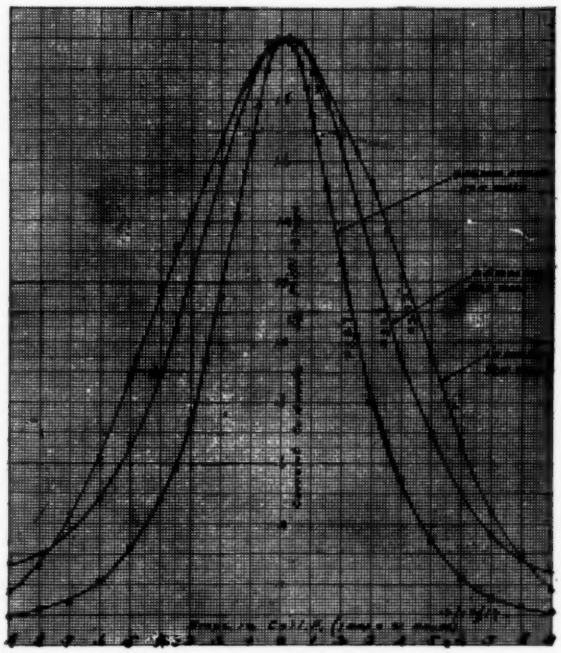

Fig. 22.

Comparison of different internal anodes at different potentials, so chosen as to give same maximum current for each.

the same for each. It is seen that even at the higher voltage required for equal current, the smaller anode is the more susceptible to magnetic field.

Conclusion.

The agreement between theory and experiment is well within the limits prescribed by the degree of symmetry of the apparatus. Experiments with more perfect apparatus are in progress by Dr. K. H. Kingdon, which, it is hoped, will show the limitations of the theory, and bring to light any new facts not hitherto taken account of.

Several useful applications have been investigated, which will be discussed elsewhere.

The above solution assumes constant e/m, and therefore applies only to potential differences of the order of 50,000 volts or less. Dr. Leigh Page has very kindly worked out the solution for the case of variable e/m, which is presented in the following paper.

Research Laboratory, General Electric Co.,
Schenectady, N. Y.,
March 3, 1921.

THEORY OF THE MOTION OF ELECTRONS BETWEEN COAXIAL CYLINDERS TAKING INTO ACCOUNT THE VARIATION OF MASS WITH VELOCITY.

BY LEIGH PAGE.

SYNOPSIS.

Theory of the Motion of Electrons between Coaxial Cylinders, the Electric Field being Radial, the Magnetic Longitudinal.—If in Hull's coaxial cylinder method of measuring e/m, large potential differences are used, it is necessary *to take account of the variation of mass with velocity.* Putting the expressions for transverse and longitudinal mass into the equations of motion and integrating we get: $e_0/m_0 = 8\pi^2c^2R^2V/(N^2 - 4\pi^2R^2V^2)$, which depends only on the larger radius R, the total magnetic flux N, and the total potential drop V, and is not affected by space charge. Because the longitudinal mass is involved, this equation provides a *method of experimentally verifying the theoretical expression for longitudinal mass.*

IN the preceding paper Dr. A. W. Hull has described a new method of measuring the ratio of charge to mass of the electron. If the difference of potential between the coaxial cylinders is sufficiently great, the electrons will acquire a velocity in the course of their motion comparable to that of light. Hence the variation of mass with velocity must be taken into account. The following is a rigorous solution of the problem insofar as concerns the term in the kinetic reaction of the electron which involves its mass, although the radiation reaction, and in fact all terms involving derivatives of the acceleration,[1] are neglected. The problem is interesting in that the solution involves both transverse and longitudinal masses. The theoretical expression for the transverse mass has been abundantly verified by Bucherer,[2] Neumann[3] and others, but so far as the author is aware, no experimental confirmation of the formula for the longitudinal mass has yet been obtained other than the rather indirect confirmation involved in Sommerfeld's[4] theory of the fine structure of spectral lines. Therefore experimental verification of the results of the theory to be presented would constitute a valuable confirmation of the theoretical formula for longitudinal mass. Furthermore the experimental data would provide a test between the symmetric electron here assumed (such as the Lorentz electron) and the asymmetric electron (such as that proposed by Parsons).

[1] See L. Page, PHYS. REV., 11, p. 376, 1918.

[2] Phys. Zeit., 9, p. 755, 1908.

[3] Ann. d. Phys., 45, p. 529, 1914.

[4] Atombau und Spektrallinien, Chap. 5, 1921.

Let

$F_r \equiv$ component of force along radius vector,
$F_\theta \equiv$ " " " " perpendicular to radius vector,
$f_r \equiv$ " " acceleration along radius vector,
$f_\theta \equiv$ " " " perpendicular to radius vector,
$\beta \equiv \frac{v}{c}$ = ratio of velocity of electron to that of light,
$m_0 \equiv$ rest mass of electron (a constant),
$e_0 \equiv$ charge on electron,
$a \equiv$ radius of inner cylinder or wire,
$R \equiv$ " " outer " ,
$E \equiv$ electric field (radial),
$H \equiv$ magnetic field (axial),
$V \equiv$ difference between the potentials of the two cylinders,
$N \equiv$ magnetic flux through the part of the cross-section intercepted between the wire and outer cylinder.

Then, parallel to the path of the electron under consideration

$$F_r\dot{r} + F_\theta r\dot{\theta} = \frac{m_0}{(1-\beta^2)^{3/2}}[f_r\dot{r} + f_\theta r\dot{\theta}],$$

and perpendicular to the path

$$F_r r\dot{\theta} - F_\theta\dot{r} = \frac{m_0}{(1-\beta^2)^{1/2}}[f_r r\dot{\theta} - f_\theta\dot{r}],$$

where, expressed in Heaviside-Lorentz rational units,

$$F_r = e_0\left[E + \frac{r\dot{\theta}}{c}H\right],$$

and

$$F_\theta = -e_0\frac{\dot{r}}{c}H.$$

Put

$$e \equiv \frac{e_0}{m_0}E,$$

$$h \equiv \frac{e_0}{m_0}H.$$

Then the equations of motion are

$$\ddot{r} - r\dot{\theta}^2 = \sqrt{1-\beta^2}\left[\left(1 - \frac{\dot{r}^2}{c^2}\right)e + \frac{r\dot{\theta}}{c}h\right], \tag{1}$$

$$\frac{d}{dt}(r^2\dot{\theta}) = -r\sqrt{1-\beta^2}\left[\frac{\dot{r}}{c}h + \frac{\dot{r}r\dot{\theta}}{c^2}e\right], \tag{2}$$

from which is easily obtained the energy equation

$$\frac{1}{2}d(\beta^2) = (1-\beta^2)^{3/2}\frac{e}{c^2}dr. \tag{3}$$

The integral of (3) is

$$\frac{1}{\sqrt{1-\beta^2}} = 1 + \frac{1}{c^2}\int_a^r e\,dr. \tag{4}$$

Substituting the value of $\sqrt{1-\beta^2}$ from (4) into (2) this equation of motion becomes

$$d(r^2\dot{\theta}) = -\frac{\frac{hr}{c}+\frac{er^2\dot{\theta}}{c^2}}{1+\frac{1}{c^2}\int_a^r e\,dr}dr, \tag{2'}$$

of which the solution is

$$r^2\dot{\theta} = -\frac{\frac{1}{c}\int_a^r h\,r\,dr}{1+\frac{1}{c^2}\int_a^r e\,dr}.$$

Now

$$\frac{\dot{r}^2}{c^2} = \beta^2 - \frac{r^2\dot{\theta}^2}{c^2}$$

$$= 1 - \frac{1+\frac{1}{c^4r^2}\left[\int_a^r h\,r\,dr\right]^2}{\left[1+\frac{1}{c^2}\int_a^r e\,dr\right]^2}.$$

The critical condition is that for which $\dot{r} = 0$ when $r = R$. But

$$\int_a^R e\,dr = \frac{e_0}{m_0}V,$$

$$\int_a^R h\,r\,dr = \frac{e_0}{m_0}\frac{N}{2\pi}.$$

Therefore

$$\frac{e_0}{m_0} = \frac{8\pi^2c^2V}{\frac{N^2}{R^2}-4\pi^2V^2}, \tag{5}$$

whereas if no account is taken of the variation of mass with velocity, (*i.e.*, if β is very small compared to unity),

$$\frac{e_0}{m_0} = \frac{8\pi^2c^2V}{\frac{N^2}{R^2}}. \tag{6}$$

It may be noted that the expression for the ratio of charge to mass involves only the total potential drop and total magnetic flux. So long as the electric field is radial, and the magnetic field has radial symmetry about the axis, it is immaterial what functions of r these fields may be. Hence the effect of space charge is completely eliminated. In electromagnetic units the expression (5) for the ratio of charge to mass becomes

$$\frac{e_0}{m_0} = \frac{8\pi^2 V}{\frac{N^2}{R^2} - 4\pi^2 \frac{V^2}{c^2}}.$$

Sloane Physics Laboratory,
Yale University.

THE TRANSMISSION OF WAVES THROUGH A SYMMETRIC OPTICAL INSTRUMENT.[1]

BY IRWIN ROMAN.

SYNOPSIS.

Gaussian Parameter Method of Studying the Transmission of a Wave Surface through a Centered System of Symmetric Lenses.—Since for each point of an incident wave surface there is a corresponding point on the refracting surface, the parameter β of the refracting surface may be expressed in terms of the parameter α of the wave surface, thus allowing the elimination of the parameter β. After expressing all quantities in terms of α, certain invariant combinations are found among the coefficients of the wave and lens surfaces, these furnishing the determination of the refracted surface. This gives a point by point correspondence between all surfaces of the system. As an application, the focal distance and longitudinal aberration are calculated and a numerical case given. The method is applicable to aspherical surfaces as simply as to spherical ones, and it is hoped that many problems in practical optics may be simplified by the present method, combined with proper choices of the parameter.

PART I. INTRODUCTION.

THE literature on the aberrations in an optical system contains a number of different methods of attack on the problems involved. Nearly all of these may be grouped into two classes, according as the chief emphasis is placed on the wave surface or the ray congruence. As a combination of the two points of view may be mentioned the methods using the characteristic function of Hamilton, which, in the hands of Bruns[2] and of Schwarzschild,[3] have yielded some valuable results. The characteristic function is essentially a relation between the coördinates of the points at the opposite ends of a particular ray segment, the results being studied without expressly finding the coördinates of one end in terms of those of the other end. The method is an implicit function method, and leads to fairly complicated analyses.

It was pointed out by Gullstrand[4] that the aberrations of an optical system could be studied more simply from the point of view of the wave surface than from that of the ray. He selected the equation of a particular wave in the form $z = f(x, y)$. So far as the present writer has been able to learn, Gullstrand has made no attempt to study the trans-

[1] Presented to The University of Chicago, in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

[2] Abh. d. math.-phys. Kl. d. Kgl. Sächs. Ges., Bd. 21, 8325 (1895).

[3] Abh. d. Kgl. Ges. d. Wiss. zu Göttingen, math.-phys. Kl. Neue Folge, IV (1905–1906).

[4] Ann. d. Phys. (4), Bd. 18, S. 941–973; also K. Svenska Vetenskapsakad. Handl., 41. No. 3 (1906–7).

formation of a wave surface, but restricts himself to the analysis of a particular surface, the aberrations being referred to that surface.

The purpose of the present investigation has been to replace the rectangular representation of the surface by a representation in terms of two Gaussian parameters such as are used in differential geometry. Let us consider a particular wave of a family incident on the surface of separation between two homogeneous and isotropic media. Suppose that the initial wave and the lens surface are determined in terms of separate parameters. At each regular point of the wave surface there is a unique normal, commonly called the ray. This ray, if effective in image formation, must pass through the lens surface, and the point of intersection determines a correspondence between the two sets of parameters. By means of this correspondence, either set of parameters may be expressed in terms of the other set, thus allowing the elimination of all but a single set. The use of this single set of parameters throughout the system, makes it possible to determine corresponding points on all surfaces of the system, since all points on a particular ray are given by the same values of the parameters throughout the entire system. It should be noted, however, that if the parameters have a simple geometric significance for one surface, they will, usually, not have a simple significance for another surface. Having determined the various parameters in terms of a common parameter, we may proceed to the study of corresponding surfaces and to the study of the transformation of the surface as the wave is transmitted through the system. Knowing the initial and the final wave of the system, we may proceed to the study of the aberrations in accordance with any desired classification or analysis.

Besides the advantage of directness in the method, there are several other advantages. Since we have a single set of parameters, we have a definite interpretation for the "order" of a quantity. While a few attempts have been made to retain powers of small quantities beyond the second, such terms are usually neglected, probably because of the complexity involved in their retention, and possibly also because there is usually a certain vagueness in the analysis. While a quantity may be of one order in terms of one parameter, it may be of a different order in terms of another parameter. A second advantage lies in the facility with which questions involving aspherical surfaces may be treated. In the usual treatments, the surfaces are assumed to be spheres or to depart only slightly from spheres.

In the preceding paragraphs, the method has been sketched for a general system. Because of the complexity which arises in carrying out the method, the investigation in this paper has been restricted to the

case in which all the surfaces are symmetrical with respect to a common axis of revolution. This allows the selection of one of the two Gaussian parameters as the azimuth around the common axis, reducing the problem to a plane problem. Besides assuming that each surface is a surface of revolution around a common axis, we shall assume that each surface and its representation is non-singular in the portions to be studied. This means that for the axis and for a certain region around the axis, there shall be no singular points of the surface or in its representation. The parameters will be selected so as to reduce to zero for the axial point. This choice of parameters is a restriction of small importance, since practically all discussions of the symmetrical system are made in terms of the optical height or of the paraxial angle. The restrictions here imposed still allow a considerable freedom in the choice of parameters to suit particular needs.

In following out the methods here discussed, we shall use power series expansion in the parameter selected, retaining all terms by means of recursion formulas. Except for the application to the first four orders, no attempt will be made to reduce these to explicit forms. The complexity increases rapidly with the order of the term, and in numerical calculations, the recursion formulas may even be more convenient than the explicit forms.

The method also makes use of invariants of two classes, parametric and optical. A parametric invariant is a quantity whose value is independent of the choice of parameter. An optical invariant is a quantity whose value is independent of which particular wave of the family is selected. The parametric invariants are geometric and are incidental rather than fundamental in the optical problem. The optical invariants, however, are fundamental, as they furnish the new surface in terms of the old surface and the interface. Unless specified to the contrary, we shall understand the single word "invariant" to refer to optical invariant.

In order to avoid possible confusion, we shall define several terms ordinarily used with varying meanings. By a refraction, we shall understand the transformation from a wave surface in one medium to that in the other, the two surfaces coinciding with the interface at the axis. By a propagation, we shall understand a transformation from one wave surface to another in the same medium. By a transmission, we shall understand a refraction between two propagations. Thus a transmission carries an arbitrary wave surface in one medium to a wave surface in the other medium at an arbitrary distance. When convenient, and where no confusion is likely to arise, the term transmission may be used for any combination of propagations and refractions, or either alone.

Part II. The Correspondence of Parameters.

As was pointed out in the introduction, there corresponds to each point on the initial wave a point on the lens surface uniquely determined as the point where a normal to the wave at the point in question cuts the lens surface. Since the system is one of revolution about a single axis; we may select that axis as the x-axis, and study the system by means of a typical meridian plane, which we may take as $z = 0$. Then the y-axis will be in the plane under consideration and normal to the axis of revolution. The symmetry assumed may be expressed in the equations of the wave surface by:

$$x = \sum_{i=0}^{\infty} A_{2i}\alpha^{2i} \qquad y = \sum_{i=0}^{\infty} B_{2i+1}\alpha^{2i+1}. \tag{1}$$

The assumption that the representation is regular at the axis is equivalent to the condition $B_1 \neq 0$. The other coefficients may be assumed to be all finite. Thus, $y = 0$ for $\alpha = 0$. Likewise, we may assume the lens surface to be given by

$$\xi = \sum_{i=0}^{\infty} C_{2i}\beta^{2i}, \qquad \eta = \sum_{i=0}^{\infty} D_{2i+1}\beta^{2i+1}, \tag{2}$$

where $D_1 \neq 0$ and all the coefficients are finite.

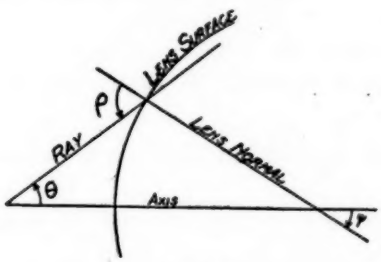

Fig. 1.

Let the normal to the lens surface make an angle ψ with the x-axis, measured as shown in the figure, positive angles being counterclockwise, as usual. Let the ray make an angle θ with the x-axis and an angle ρ with the lens surface normal. Then, since the ray is normal to the wave, it follows that

$$\tan\theta = -\frac{dx}{d\alpha}\Big/\frac{dy}{d\alpha} \qquad \tan\psi = -\frac{d\xi}{d\beta}\Big/\frac{d\eta}{d\beta}. \tag{3}$$

If the point (ξ, η) of the lens surface corresponds to the point (x, y) of the wave surface, then it follows that

$$(\eta - y)/(\xi - x) = \tan\theta = -\frac{dx}{d\alpha}\Big/\frac{dy}{d\alpha}$$

or

$$(\eta - y)\frac{dy}{d\alpha} + (\xi - x)\frac{dx}{d\alpha} = 0. \tag{4}$$

Equation (4) represents the correspondence between α and β and enables us to determine E_i where

$$\beta = \sum_{i=0}^{\infty} E_i \alpha^i. \tag{5}$$

From equation (1), we have

$$\left.\begin{aligned} \frac{dx}{d\alpha} &= \sum_{i=0}^{\infty} (2i+2) A_{2i+2} \alpha^{2i+1}, \\ \frac{dy}{d\alpha} &= \sum_{i=0}^{\infty} (2i+1) B_{2i+1} \alpha^{2i}. \end{aligned}\right\} \tag{6}$$

For $\alpha = 0$, $y = dx/d\alpha = 0$ and $dy/d\alpha = B_1 \neq 0$, so that equation (4) requires that $\eta = 0$ for $\alpha = 0$ or that the solution we want is the one for which $E_0 = 0$, giving

$$\beta = \sum_{i=1}^{\infty} E_i \alpha^i. \tag{7}$$

To express ξ and η in terms of α, we need the various powers of β in terms of α. For this purpose, let

$$\beta^k = \sum_{j=0}^{\infty} I_j^k \alpha^{j+k}. \tag{8}$$

Inspection of equations (7) and (8) shows that I_j^k has the following defining properties:

(*a*) I_j^k is the sum of terms each involving k of the E_i, one from each of the factors β.

(*b*) Since the order, or subscript, of each E_i denotes the power of α that it accompanies from β, the sum of the subscripts in each term of I_j^k is $(j + k)$.

(*c*) The numerical coefficient of each term in I_j^k is the number of distinct permutations possible among the subscripts of that term. If E_i occurs n_i times in a particular term, this number is

$$\frac{k!}{\Pi_i(n_i!)} \quad \text{where} \quad \sum_i n_i = k \quad \text{and} \quad \sum_i i n_i = j + k.$$

Hence, we have

$$I_j^k = \sum_u \frac{k!}{\Pi_i(n_i!)} \prod_i (E_i^{n_i}), \tag{9}$$

where $\sum_i n_i = k$ and $\sum_i i n_i = j + k$ while $\sum_u$ extends over all possible partitions of $(j + k)$ things into k parts.

In particular, I_j^k has the following useful properties:

(*A*) The highest order E_i among all the terms of I_j^k is in $E_1^{k-1}E_{j+1}$.

(*B*) If j is odd, each term of I_j^k must have at least one E_i of even order. This follows from the fact that if each E_i were of odd order, the sum would be even or odd with k, while $(j+k)$ is the opposite.

(*C*) $I_0^0 = 1$, $I_j^0 = 0$ for $j > 0$, $I_j^1 = E_{j+1}$.

Inserting equations (8) in equations (2), we get

$$\xi = \sum_{i,j=0}^{\infty} C_{2i} I_j^{2i} \alpha^{2i+j}, \qquad \eta = \sum_{i,j=0}^{\infty} D_{2i+1} I_j^{2i+1} \alpha^{2i+j+1}. \tag{10}$$

By means of equations (1), (6) and (10), equation (4) becomes

$$\sum_{i,j,k=0}^{\infty} \left\{ \begin{aligned} & C_{2i} I_j^{2i} (2k+2) A_{2k+2} \alpha^{2i+j+2k+1} \\ & \quad - (2j+2) A_{2j+2} A_{2i} \alpha^{2i+2j+1} \\ & \quad + D_{2i+1} I_j^{2i+1} (2k+1) B_{2k+1} \alpha^{2i+j+2k+1} \\ & \quad - (2j+1) B_{2j+1} B_{2i+1} \alpha^{2i+2j+1} \end{aligned} \right\} = 0. \tag{11}$$

The coefficient of α^{2m} is

$$\sum_{k=0}^{m-1} \sum_{i=0}^{m-k-1} [(2k+2) A_{2k+2} C_{2i} I_{2m-2k-2i-1}^{2i} + (2k+1) B_{2k+1} D_{2i+1} I_{2m-2k-2i-1}^{2i+1}] = 0. \tag{12}$$

Since $j = 2m - 2k - 2i - 1$ is odd, each term in equation (12) contains an E_i of even order. Since $I_{2m-1}^0 = 0$, the highest order E_i occurs for $k = i = 0$ and is E_{2m}, the term being $B_1 D_1 I_{2m-1}^1 = B_1 D_1 E_{2m}$. Hence by (12), E_{2m} is the sum of terms each of which involves a factor E_i of even order below $2m$. In particular, for $m = 1$, we have, by equation (12), $k = i = 0$ so

$$2A_2 C_0 I_1^0 + B_1 D_1 I_1^1 = B_1 D_1 E_2 = 0. \tag{13}$$

Since B_1 and D_1 were selected different from zero, it follows that $E_2 = 0$. Further, we have

$$E_{2m} = 0 \qquad \text{and} \qquad I_{2j+1}^k = 0. \tag{14}$$

As a consequence of equations (14), we may write equations (8) and (9) as follows:

$$\left. \begin{aligned} \beta^k &= \sum_{j=0}^{\infty} I_{2j}^k \alpha^{2j+k}, \text{ where} \\ I_{2j}^k &= \sum_u \frac{k!}{\Pi_i (n_i!)} \prod_i E_{2i+1}^{n_i}, \\ & \sum_i n_i = k, \\ & \sum_i (2i+1) n_i = k + 2j \end{aligned} \right\} \tag{15}$$

and where $\sum_u$ extends over all partitions of $(k+2j)$ into k parts.

By means of equations (14), equation (11) becomes

$$\sum_{i,j,k=0}^{\infty} \left\{ \begin{array}{l} [(2k+2)A_{2k+2}C_{2i}{I_{2j}}^{2i} \\ \qquad + (2k+1)B_{2k+1}D_{2i+1}{I_{2j}}^{2i+1}]\alpha^{2i+2j+2k+1} \\ \qquad - [(2j+2)A_{2i}A_{2j+2} \\ \qquad + (2j+1)B_{2i+1}B_{2j+1}]\alpha^{2i+2j+1} \end{array} \right\} = 0. \quad (16)$$

The coefficient of α^{2m+1} is

$$\left. \begin{array}{l} \sum_{k=0}^{m}\sum_{i=0}^{m-k}[(2k+2)A_{2k+2}C_{2i}I_{2m-2k-2i}^{2i} \\ \qquad + (2k+1)B_{2k+1}D_{2i+1}I_{2m-2k-2i}^{2i+1}] \\ \quad - \sum_{i=0}^{m}[(2i+2)A_{2i+2}A_{2m-2i} + (2i+1)B_{2i+1}B_{2m-2i+1}] \end{array} \right\} = 0. \quad (17)$$

Since

$$\sum_{k=0}^{m}\sum_{i=0}^{m-k}\varphi(i,k) = \varphi(0,0) + \sum_{i=1}^{m}\varphi(i,0) + \sum_{k=1}^{m}\sum_{i=0}^{m-k}\varphi(i,k),$$

equation (17) becomes

$$\left\{ \begin{array}{l} [2A_2C_0{I_{2m}}^0 + B_1D_1{I_{2m}}^1] \\ \quad + \sum_{i=1}^{m}[2A_2C_{2i}I_{2m-2i}^{2i} + B_1D_{2i+1}I_{2m-2i}^{2i+1}] \\ \quad + \sum_{k=1}^{m}\sum_{i=0}^{m-k}[(2k+2)A_{2k+2}C_{2i}I_{2m-2k-2i}^{2i} \\ \quad + (2k+1)B_{2k+1}D_{2i+1}I_{2m-2k-2i}^{2i+1}] \\ \quad - \sum_{i=0}^{m}[(2i+2)A_{2i+2}A_{2m-2i} + (2i+1)B_{2i+1}B_{2m-2i+1}] \end{array} \right\} = 0. \quad (18)$$

Since ${I_{2m}}^1 = E_{2m+1}$, this gives the recursion formula:

$$E_{2m+1} = \frac{1}{B_1D_1}\left\{ \begin{array}{l} \sum_{i=0}^{m}[(2i+2)A_{2i+2}A_{2m-2i} + (2i+1)B_{2i+1}B_{2m-2i+1}] \\ - 2A_2C_0{I_{2m}}^0 - \sum_{i=1}^{m}[2A_2C_{2i}I_{2m-2i}^{2i} \\ \qquad + B_1D_{2i+1}I_{2m-2i}^{2i+1}] \\ - \sum_{k=1}^{m}\sum_{i=0}^{m-k}[(2k+2)A_{2k+2}C_{2i}I_{2m-2k-2i}^{2i} \\ \qquad + (2k+1)B_{2k+1}D_{2i+1}I_{2m-2k-2i}^{2i+1}] \end{array} \right\} \quad (19)$$

Equation (19) furnishes E_{2m+1}, after which equations (15) furnish ${I_{2j}}^k$ so that substitution in equation (10) gives:

$$\left.\begin{aligned}\xi &= \sum_{m=0}^{\infty} F_{2m}\alpha^{2m} \\ \eta &= \sum_{m=0}^{\infty} G_{2m+1}\alpha^{2m+1}\end{aligned}\right\} \text{where} \left\{\begin{aligned}F_{2m} &= \sum_{i=0}^{m} C_{2i} I_{2m-2i}^{2i} \\ G_{2m+1} &= \sum_{i=0}^{m} D_{2i+1} I_{2m-2i}^{2i+1}\end{aligned}\right\}. \quad (20)$$

While equations (20) give the value of G_{2m+1}, it may be expressed in a more convenient form as follows. By equations (1), (4), (6), and (20), we have

$$\sum_{i,j=0}^{\infty} [(2i+1)(B_{2j+1} - G_{2j+1})B_{2i+1} + (2i+2)(A_{2j} - F_{2j})A_{2i+2}]\alpha^{2i+2j+1} = 0. \quad (21)$$

The coefficient of α^{2m+1} is

$$\sum_{j=0}^{m} [(2m-2j+1)(B_{2j+1} - G_{2j+1})B_{2m-2j+1} + (2m-2j+2)(A_{2j} - F_{2j})A_{2m-2j+2}] = 0. \quad (22)$$

Since $\sum_{j=0}^{m} \varphi(j) = \varphi(m) + \sum_{j=0}^{m-1} \varphi(j)$, we have from equation (22)

$$\left.\begin{aligned}G_{2m+1} = B_{2m+1} &+ \frac{1}{B_1}\sum_{j=0}^{m-1}(2m-2j+1)(B_{2j+1} - G_{2j+1})B_{2m-2j+1} \\ &+ \frac{1}{B_1}\sum_{j=0}^{m}(2m-2j+2)(A_{2j} - F_{2j})A_{2m-2j+2}\end{aligned}\right\}. \quad (23)$$

We have thus expressed in β terms of α, and are able to eliminate the parameter β from the problem, expressing everything in terms of the parameter α. We have also found explicit values for the coördinates of each point of the lens surface in terms of the value of the parameter specifying the corresponding point on the incident wave surface. Since the coefficients F_{2m} and G_{2m+1} are for the lens surface in terms of α, it follows that they are independent of the choice of β. Thus, when F_{2m} and G_{2m+1} are expressed in terms of A_{2m}, B_{2m+1}, C_{2m} and D_{2m+1}, the coefficients of A_{2m} and B_{2m+1} will be combinations of the C_{2m} and D_{2m+1} which will be the same for all parameters β. Hence these give parametric invariants of the curve which describes the lens surface and hence of all curves with the assumed type of symmetry. The details of this will appear more clearly in the special cases treated in Part IV.

PART III. TRANSFORMATION OF THE WAVE SURFACE.

Having expressed the coördinates of the lens surface in terms of the parameter of the incident wave, we may now proceed at once to the study of how the wave is transformed in its passage through the system. We shall assume that the incident wave

$$x' = \sum_{i=0}^{\infty} A'_{2i}\alpha^{2i} \qquad y' = \sum_{i=0}^{\infty} B'_{2i+1}\alpha^{2i+1} \tag{1'}$$

is propagated to the lens surface

$$\xi = \sum_{i=0}^{\infty} F_{2i}\alpha^{2i} \qquad \eta = \sum_{i=0}^{\infty} G_{2i+1}\alpha^{2i+1}, \tag{20}$$

is there refracted in the usual way and is finally propagated into the wave

$$x'' = \sum_{i=0}^{\infty} A''_{2i}\alpha^{2i} \qquad y'' = \sum_{i=0}^{\infty} B''_{2i+1}\alpha^{2i+1}. \tag{1''}$$

We shall thus want to determine A''_{2i} and B''_{2i+1} in terms of A'_{2i}, B'_{2i+1}, F_{2i}, G_{2i+1} and A_0'', the latter quantity specifying which wave of the refracted family we are considering. The same value of α specifies corresponding points (x', y'), (ξ, η) and (x'', y'').

The law of refraction is equivalent to the optical invariance of $\mu \sin \rho$ where μ is the index of refraction of the medium and ρ is the angle between the normals to the lens surface and the wave surface, respectively. If we use primes to refer to the first medium and double primes to refer to the second medium, this may be stated in the form $\mu' \sin \rho' = \mu'' \sin \rho''$. From the figure, we have

$$\sin \rho = \sin (\theta - \psi) = \sin \theta \cos \psi - \cos \theta \sin \psi. \tag{24}$$

By means of equations (3) and (24) and setting

$$\mu \sin \rho = \sum_{i=0}^{\infty} L_{2i+1}\alpha^{2i+1} \tag{25}$$

we find that

$$\left.\begin{aligned}
\sum_{i=0}^{\infty} L_{2i+1}\alpha^{2i+1} &= \frac{\mu \sum_{i=0}^{\infty} M_{2i+1}\alpha^{2i+1}}{\sum_{j=0}^{\infty} H_{2j}\alpha^{2j} \sum_{k=0}^{\infty} J_{2k}\alpha^{2k}} \quad \text{where} \\
\sum_{i=0}^{\infty} M_{2i+1}\alpha^{2i+1} &= \frac{d\xi}{d\alpha}\frac{dy}{d\alpha} - \frac{dx}{d\alpha}\frac{d\eta}{d\alpha} \\
\sum_{j=0}^{\infty} H_{2j}\alpha^{2j} &= \sqrt{\left(\frac{dx}{d\alpha}\right)^2 + \left(\frac{dy}{d\alpha}\right)^2} \\
\sum_{k=0}^{\infty} J_{2k}\alpha^{2k} &= \sqrt{\left(\frac{d\xi}{d\alpha}\right)^2 + \left(\frac{d\eta}{d\alpha}\right)^2}
\end{aligned}\right\} \tag{26}$$

By means of equations (1), (20) and (26), after a few simple reductions,

we find

$$\left.\begin{aligned} &H_0 = B_1, \qquad J_0 = G_1, \qquad M_1 = 2(F_2B_1 - A_2G_1), \\ &\qquad\qquad\qquad\qquad\qquad\qquad L_1 = \frac{2\mu}{B_1G_1}(F_2B_1 - A_2G_1) \end{aligned}\right\}$$

while for $m > 0$, we get

$$\left.\begin{aligned} H_{2m} &= \frac{1}{2B_1}\left\{\sum_{i=0}^{m-1}(2i+2)(2m-2i)A_{2m-2i}A_{2i+2}\right. \\ &\qquad + \sum_{i=0}^{m}(2i+1)(2m-2i+1)B_{2i+1}B_{2m-2i+1} \\ &\qquad \left. - \sum_{i=1}^{m-1}H_{2i}H_{2m-2i}\right\} \\ J_{2m} &= \frac{1}{2G_1}\left\{\sum_{i=0}^{m-1}(2i+2)(2m-2i)F_{2m-2i}F_{2i+2}\right. \\ &\qquad + \sum_{i=0}^{m}(2i+1)(2m-2i+1)G_{2i+1}G_{2m-2i+1} \\ &\qquad \left. - \sum_{i=1}^{m-1}J_{2i}J_{2m-2i}\right\} \\ M_{2m+1} &= \sum_{i=0}^{m}(2i+2)(2m-2i+1)(F_{2i+2}B_{2m-2i+1} \\ &\qquad\qquad - A_{2i+2}G_{2m-2i+1}) \\ L_{2m+1} &= \frac{1}{B_1G_1}\left\{\mu M_{2m+1} - \sum_{i=0}^{m-1}L_{2i+1}\sum_{j=0}^{m-i}H_{2j}J_{2m-2i-2j}\right\} \end{aligned}\right\} \quad (27)$$

If we define

$$S_{2m+1} = \frac{1}{G_1}\sum_{i=0}^{m}L_{2i+1}J_{2m-2i} = L_{2m+1} + \frac{1}{G_1}\sum_{i=0}^{m-1}L_{2i+1}J_{2m-2i}, \qquad (28)$$

we get

$$S_{2m+1} = \frac{1}{B_1G_1}\left\{\mu M_{2m+1} - \sum_{i=0}^{m-1}L_{2i+1}\sum_{j=1}^{m-i}H_{2j}J_{2m-2i-2j}\right\}. \qquad (29)$$

The quantities F_{2m}, G_{2m+1}, J_{2m}, L_{2m+1} and S_{2m+1} are optical invariants and have the same value in the two sets of primed symbols, *i.e.*, in the two media. The invariants G_{2m+1} and S_{2m+1} involve B_{2m+1} and A_{2m+2} along with terms of lower order and F_{2m} and G_{2m+1}, which are invariant. Hence the equations

$$G'_{2m+1} = G''_{2m+1} \quad \text{and} \quad S'_{2m+1} = S''_{2m+1}, \qquad (30)$$

which are linear in B_{2m+1} and A_{2m+2} if $m > 0$, may be solved for B''_{2m+1} and A''_{2m+2}. In the case $m = 0$, the equations are not linear, but, as shown in Part IV., the solution may be effected without difficulty. The invariant L_{2m+1} might be used instead of S_{2m+1}, but the latter is simpler. While S_{2m+1} involves G_{2m+1}, this may be eliminated by means of equation (23).

The solution for A''_{2m+2} and B''_{2m+1} completes the problem of finding the new wave in terms of the old wave and the refracting surface. The solution of equations (30) becomes increasingly complex as m increases.

PART IV. TERMS OF THE FIRST FOUR ORDERS.

To illustrate the use of the preceding formulas, we shall calculate the results for the cases $m = 0$ and $m = 1$. The case $m = 0$ is the elementary Gaussian theory, while the case $m = 1$ is essentially the theory of first order aberrations, so far as these aberrations have a direct interpretation in the case of a system of revolution. For the case of $m = 0$, we shall calculate the case of a general transmission, but for the case of $m = 1$, we shall calculate the cases of propagation and refraction separately, because of the complexity of the formulas. As a matter of notation, let

$$l' = F_0 - A_0', \quad l'' = F_0 - A_0'', \quad \text{and} \quad l = l' - l'' = A_0'' - A_0'. \tag{31}$$

Case I. Gaussian Theory. $m = 0$.—By substitution in the various formulas (23), (27), (29) and (30), we get

$$\left.\begin{aligned} G_1' = B_1' - 2l'(A_2'/B_1') = G_1'' = B_1'' - 2l''(A_2''/B_1'') \\ S_1' = L_1' = 2\mu'\left(\frac{F_2}{G_1} - \frac{A_2'}{B_1'}\right) = S_1'' = 2\mu''\left(\frac{F_2}{G_1} - \frac{A_2''}{B_1''}\right) \end{aligned}\right\}. \tag{32}$$

$$\left.\begin{aligned} B_1'' - 2l''(A_2''/B_1'') = B_1' - 2l'(A_2'/B_1') \\ (A_2''/B_1'') = \frac{\mu'}{\mu''}\left(\frac{A_2'}{B_1'}\right) + \frac{F_2(\mu'' - \mu')}{G_1\mu''} \end{aligned}\right\}. \tag{33}$$

Solving for B_1'' and (A_2''/B_1''), we get

$$\left.\begin{aligned} B_1'' = B_1' - \frac{2A_2'(l'\mu'' - l''\mu')}{B_1'\mu''} + \frac{2l''F_2(\mu'' - \mu')}{G_1\mu''} \\ A_2'' = \left[\frac{\mu'A_2'}{\mu''B_1'} + \frac{F_2(\mu'' - \mu')}{G_1\mu''}\right]B_1'' \end{aligned}\right\}. \tag{34}$$

For refraction, $l' = l'' = 0$ so that

$$B_1'' = B_1' = G_1 \qquad \text{and} \qquad A_2'' = [\mu'A_2' + F_2(\mu'' - \mu')]/\mu''. \tag{34r}$$

For propagation, $\mu' = \mu'' = 1$ so that

$$B_1'' = B_1' - 2l(A_2'/B_1') \qquad A_2'' = A_2' - 2l(A_2'/B_1')^2 \tag{34p}$$

Case II. First Order Aberrations. $m = 1$.—In this case, we have,

by equations (23), (27) and (29),

$$\left.\begin{aligned} G_3 &= B_3 + [3(B_1 - G_1)B_3 + 4(A_0 - F_0)A_4 + 2(A_2 - F_2)A_2]/B_1 \\ H_2 &= 3B_3 + (2A_2^2/B_1) \\ M_3 &= 6(F_2B_3 - A_2G_3) + 4(F_4B_1 - A_4G_1) \\ S_3 &= (\mu M_3/B_1G_1) - (L_1H_2/B_1) \end{aligned}\right\} \quad (35)$$

For refraction, $A_0'' = A_0' = F_0$, $B_1'' = B_1' = G_1$, so that

$$\left.\begin{aligned} G_3 &= B_3' + 2A_2'(A_2' - F_2)/B_1' = B_3'' + 2A_2''(A_2'' - F_2)/B_1'' \\ L_1 &= 2\mu(F_2 - A_2)/B_1 \\ M_3 &= (F_2 - A_2)(6B_1B_3 + 12A_2^2)/B_1 + 4B_1(F_4 - A_4) \\ S_3 &= \frac{4\mu'}{B_1'}\left[(F_4 - A_4') + \frac{2A_2'^2}{B_1'^2}(F_2 - A_2')\right] \\ &= \frac{4\mu''}{B_1''}\left[(F_4 - A_4'') + \frac{2A_2''^2}{B_1''^2}(F_2 - A_2'')\right] \end{aligned}\right\} \quad (36r)$$

$$\left.\begin{aligned} B_3'' &= B_3' + \frac{2(A_2' - F_2)}{B_1'}\left(A_2' - \frac{\mu'}{\mu''}A_2''\right) \\ A_4'' &= F_4 + \frac{\mu'}{\mu''}(A_4' - F_4) + \frac{2\mu'}{B_1'^2\mu''}(A_2' - F_2)(A_2'^2 - A_2''^2) \end{aligned}\right\} \quad (37r)$$

A_2'' is known by equations (34r). For propagation, we may select the initial wave as coinciding with the refracting surface, so that $A_{2k}' = F_{2k}$ and $B_{2k+1}' = G_{2k+1}$. Then, by equations (34p),

$$B_1' - B_1'' = G_1 - B_1'' = 2lA_2'/B_1', \qquad A_2' - A_2'' = 2lA_2'^2/B_1'^2,$$
$$A_2''/B_1'' = A_2'/B_1'.$$

Hence

$$\left.\begin{aligned} &\left(1 - \frac{6A_2'l}{B_1'B_1''}\right)B_3'' + \frac{4l}{B_1''}A_4'' = B_3' + 4l\left(\frac{A_2'}{B_1'}\right)^3, \\ &L_1'' = 2\mu''\left(\frac{A_2'}{B_1'} - \frac{A_2''}{B_1''}\right) = 0, \\ &M_3'' = 6(A_2'B_3'' - A_2''B_3') + 4(A_4'B_1'' - A_4''B_1'), \\ &S_3'' = \frac{\mu''}{B_1'B_1''}\{6(A_2'B_3'' - A_2''B_3') + 4(A_4'B_1'' - A_4''B_1')\} = 0, \\ &3A_2'B_3'' - 2B_1'A_4'' = 3A_2''B_3' - 2A_4'B_1''. \end{aligned}\right\} \quad (36p)$$

Solving the first and last of equations (36p), we get

$$\left.\begin{aligned} B_3'' &= B_3' + l\left[\frac{6A_2'B_3'}{B_1'^2} - \frac{4A_4'}{B_1'} + 4\left(\frac{A_2'}{B_1'}\right)^3\right], \\ A_4'' &= A_4' + l\left[6\left(\frac{A_2'}{B_1'}\right)^4 + 12\left(\frac{B_3'}{B_1'}\right)\left(\frac{A_2'}{B_1'}\right)^2 - 8\left(\frac{A_2'}{B_1'}\right)\left(\frac{A_4'}{B_1'}\right)\right]. \end{aligned}\right\} \quad (37p)$$

The calculations for F_0, F_2 and F_4 are made without special difficulty and the results are as follows:

$$\left.\begin{aligned} &F_0 = C_0 I_0^0 = C_0 \qquad F_2 = C_0 I_2^0 + C_2 I_0^2 = C_2 E_1^2, \\ &F_4 = C_0 I_4^0 + C_2 I_2^2 + C_4 I_0^4 = 2C_2 E_1 E_3 + C_4 E_1^4, \\ &E_1 = \frac{B_1}{D_1} + \frac{2A_2(A_0 - C_0)}{B_1 D_1}, \\ &E_3 = \frac{2A_2^2}{B_1 D_1} + \frac{4B_3}{D_1} - \frac{2A_2 C_2 E_1^2}{B_1 D_1} - \frac{D_3 E_1^3}{D_1} + \frac{4A_4(A_0 - C_0)}{B_1 D_1} \\ &\qquad\qquad - \frac{3B_3 E_1}{B_1}. \end{aligned}\right\} \quad (38)$$

For pure refraction, $C_0 = A_0$ so that equations (38) become

$$\left.\begin{aligned} &E_1 = \frac{B_1}{D_1}, \qquad E_3 = \frac{2A_2^2}{B_1 D_1} + \frac{B_3}{D_1} - \frac{2A_2 C_2 B_1}{D_1^3} - \frac{D_3 B_1^3}{D_1^4}, \\ &F_0 = A_0, \qquad F_2 = \frac{C_2 B_1^2}{D_1^2}, \\ &F_4 = \left(\frac{C_2}{D_1^2}\right)(4A_2^2 + 2B_1 B_3) - 4A_2 B_1^2\left(\frac{C_2}{D_1^2}\right)^2 \\ &\qquad\qquad - B_1^4\left(\frac{2C_2 D_3 - C_4 D_1}{D_1^5}\right). \end{aligned}\right\} \quad (38r)$$

As was noted above (see end of part III.), the coefficients of the wave surface symbols are combinations of the lens surface coefficients which are invariant under a transformation of the parameter β of the type assumed. Hence we have two parametric invariants $P_2 = C_2/D_1^2$ and $P_4 = (C_4 D_1 - 2C_2 D_3)/D_1^5$. That these are actually parametric invariants may be verified by direct calculation, assuming that $\beta = a_1\gamma + a_3\gamma^3$. While it may appear that the assumption of pure refraction is a loss of generality, such is not the case. The values for F_2 and F_4 using the general values in equations (38) are quite complicated and lead to the same values of P_2 and P_4 as given above.

PART V. CONCLUDING REMARKS.

While no exhaustive analysis of the preceding results has been made, several conclusions may be mentioned.

The invariant S_{2k+1} may be considered as a generalization of the so-called "zero invariant." Since the axial curvature of the wave is $K = 2A_2/B_1^2$ and of the lens is $k = 2F_2/G_1^2$, the invariant S_1 is

$$S_1 = \mu\left(\frac{2F_2}{G_1} - \frac{2A_2}{B_1}\right) = \mu(kG_1 - KB_1).$$

For pure refraction, $G_1 = B_1$, so that we get the usual zero invariant

$$Q = S_1/G_1 = \mu(k - K). \tag{39}$$

While the assumption that the entire system is one of revolution makes a full analysis of the image formation unsatisfactory, we may nevertheless calculate the longitudinal aberration. Let the normal to the wave at the point (x, y) cut the axis at the point $(X, 0)$. Then X will vary with α, unless the surface is spherical. If we set

$$X = \sum_{m=0}^{\infty} \lambda_{2m}\alpha^{2m}, \tag{40}$$

λ_{2m} will be the longitudinal aberration of order $2m$. It may be shown without difficulty, by the preceding methods, that

$$\lambda_{2m} = \frac{1}{A_2}\left[\frac{1}{2}\sum_{i=0}^{m}(2i+1)B_{2i+1}B_{2m-2i+1} + \sum_{i=0}^{m}(i+1)A_{2i+2}A_{2m-2i} - \sum_{i=1}^{m}(i+1)A_{2i+2}\lambda_{2m-2i}\right]. \tag{41}$$

For $m = 0$, we have $\lambda_0 = A_0 + B_1^2/2A_2$. Since the value of X reduces to λ_0 for $\alpha = 0$, and since the wave surface cuts the axis at A_0, the quantity $B_1^2/2A_2$ represents the axial distance from the wave to the cusp of the caustic. If the initial wave is plane, this distance reduces to the ordinary focal length. The ordinary value of the longitudinal aberration (the so-called spherical aberration) is

$$\lambda_2 = A_2 + 2B_1B_3/A_2 - A_4(B_1/A_2)^2.$$

The results calculated by this method agree very satisfactorily with the results obtained by the usual methods. It may be mentioned that in this method the emphasis is placed on the wave and not on the aberrations. In order to study the aberrations and the effects of the transmission on them, we have merely to calculate the aberrations for the initial and for the final surfaces of the wave family and to subtract the one from the other.

As an example of the agreement between the present method and the usual one, let us consider a particular telescopic system as follows. Let the front lens have a front curvature of 2.0538 and a rear curvature of

-3.0538. Let the rear lens have a front curvature of -3.0538 and a rear curvature of -0.4394. Let the indices of refraction of the front and rear lenses be 1.51115 and 1.61624, respectively, with respect to air, the system being used in air. Let the lenses have axial thicknesses of 0.0065 and 0.0050, respectively, the axial separation being 0.0005. Let the aperture be 1/30. Then the present method gives the following results, for a plane incident wave:

	B_1	A_2	B_2	A_4
Approaching first surface.....	1.	0.	0.	0.
Leaving " "	1.	0.34735	0.47208	0.53027
Approaching second "	0.99548	0.34578	0.46578	0.52570
Leaving " "	0.99548	1.29596	−5.55684	−3.53208
Approaching third "	0.99418	1.29427	−5.56713	−3.56175
Leaving " "	0.99418	0.22537	0.94578	0.16635
Approaching fourth "	0.99191	0.22486	0.94914	0.16785
Leaving " "	0.99191	0.49663	0.43533	0.30213

The focal point is 0.99057 beyond the axial point of the last surface. The usual theory gives this value as 0.99059. The longitudinal aberration here is 1.03034 per unit aperture or 0.0011448, the usual theory giving 0.0011446.

While some of the formulas have been given for a refraction between two propagations, it is usually more convenient to treat the cases of refraction and of propagation separately, especially in numerical calculations. The formulas so divided are readily adapted to the use of a calculating machine, although not to the use of logarithms.

As a particular case,[1] let us consider the case of a plane wave refracted through a single spherical surface of curvature k, and let us write μ for μ'/μ''. Let us take the optical height as parametric. Then

$$\left.\begin{aligned} x &= a, \\ y &= \alpha, \\ \xi &= a + \tfrac{1}{2}k\alpha^2 + \tfrac{1}{8}k^3\alpha^4, \\ \eta &= \alpha, \end{aligned}\right\} \qquad \left\{\begin{aligned} A_0 &= C_0 = a, \\ A_2 &= A_4 = B_3 = D_3 = 0, \\ B_1 &= D_1 = 1, \\ C_2 &= \tfrac{1}{2}k, \\ C_4 &= \tfrac{1}{8}k^3, \end{aligned}\right.$$

$$\left.\begin{aligned} \frac{C_2}{D_1^2} &= \frac{1}{2}k, \\ \frac{C^4}{D_1^4} - \frac{2C_2D_3}{D_1^5} &= \frac{1}{8}k^3, \end{aligned}\right\} \qquad \left\{\begin{aligned} F_0 &= C_0, \\ F_2 &= \tfrac{1}{2}k, \\ F_4 &= \tfrac{1}{8}k^3. \end{aligned}\right.$$

[1] See, *e.g.*, Preston, Theory of Light, § 73.

The refracted wave has

$$B_1 = 1,$$
$$A_2 = \tfrac{1}{2}k(1 - \mu),$$
$$B_3 = \tfrac{1}{2}k^2\mu(1 - \mu),$$
$$A_4 = \tfrac{1}{8}k^3(1 - \mu)(1 + 2\mu - 2\mu^2),$$

The second order longitudinal aberration is

$$\lambda_2 = \frac{k\mu^2}{2(\mu - 1)},$$

This shows that if the optically denser medium is on the concave side of the refracting surface, the surface is over-corrected for aberration, for λ_2 is positive which means that the cusp of the caustic is towards the lens surface. The focal length of the surface is $B_1^2/2A_2 = 1/k(1 - \mu)$.

While no attempt has been made to apply the foregoing results to the problems of practical optics, it is to be hoped that these problems may be materially simplified by the proper choice of parameters to meet the demands of each particular case. Since the geometrical significance of the parameter is of no essential value, there is more freedom in the choice of parameters than is customary in most treatments of geometrical optics. To review what has been said, the use of a single parameter, besides the azimuth, clarifies the interpretation of orders of small quantities. The recursion formulas, furnish terms of all orders, to be used when occasion demands. Since the sphericity of the lens surfaces has no essential part in the method, it follows that the method is directly applicable to the cases of aspherical lenses. Besides furnishing the final surface as a whole, the present method gives a point by point correspondence between all the surfaces of the system. While many of the quantities involved become indeterminate at a focal point, this furnishes no special difficulty in tracing a surface through a focal point.

In closing, the writer wishes to thank Professor Arthur C. Lunn, who suggested the present investigation and who made numerous helpful suggestions during its development.

AN EXTENSION OF THE RANGE OF THE McLEOD GAUGE.

BY A. H. PFUND.

SYNOPSIS.[1]

Modification of the McLeod Gauge.—In order to extend the range of a McLeod gauge, the pressure of the gas which is forced into the capillary is measured by means of *a hot-wire gauge.* Since the ratio of the compression is known, it is possible to determine the true pressure. The hot-wire gauge consists of a loop of tungsten wire, sealed into the closed end of the capillary and connected so as to form part of a Wheatstone bridge. The hot wire gauge is calibrated directly against the McLeod gauge. Because of the fact that the cooling effect varies with the character of the gas, a separate calibration for each gas is necessary.

Tests Made on the Gauge with Air.—Careful tests prove that the results obtained for air, the only gas used, are reliable. *The limit of the hot-wire McLeod gauge* was found to be 1.7×10^{-7} mm. hg. or 2.25×10^{-4} bar, while that of the unaided McLeod gauge was found to be 5.26×10^{-3} mm. Hg. The range of the McLeod gauge is therefore extended three hundred fold. By using a sensitive wall galvanometer in place of the portable galvanometer, a sensibility of 3.4×10^{-10} mm. or 4.5×10^{-1} bar was realized.

THE sensibility limit of a McLeod gauge has been reached when the length of the trapped air column and the difference in level of the two mercury columns are of the order of magnitude of 1 mm. Under these conditions, however, the trapped air is subjected to the pressure of 1 mm. of merucry—which is large from the standpoint of high vacua. If it were possible to measure the pressure of this trapped air, then, knowing the ratio of compression, it would be possible to extend greatly the range of the McLeod gauge.

This idea is realized by sealing a fine loop of tungsten wire into the top of the closed capillary. Using this arrangement (which is the well-known "hot-wire" gauge) in conjunction with a Wheatstone Bridge and a portable galvanometer, it is possible to extend the range of the McLeod gauge about 300-fold.

DESCRIPTION OF APPARATUS.

The apparatus used is shown in Fig. 1. This particular form of the gauge was adopted because of the convenience of baking out the glass. This is accomplished by inverting a small electric furnace over the top of the gauge. The filament (f) was made of 2 mil tungsten wire and had an over-all length of 6 mm. The resistances r_1, r_2, r_3, forming part of a simple Wheatstone bridge, were made of No. 22 chromel wire. While

[1] Presented at Washington meeting of Am. Phys. Soc. (1920).

no definite information can be given as to the highest temperature reached by the tungsten filament in a high vacuum, it may be stated that when the current flowing through the filament was increased to 130 m.a. the tip of the filament appeared faintly red in a darkened room. The filament current actually used was about 100 m.a. For several reasons

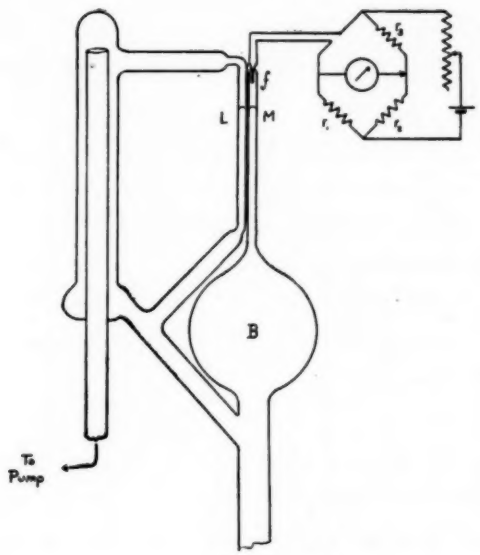

Fig. 1.

the bore of the capillary was chosen large: 2.0 mm. Not only is the range of the gauge on the high-pressure side increased but such defects as mercury sticking to the walls, negative pressures, etc,. are reduced to a minimum. Careful tests showed that no measurable amount of gas was caught between the rising column of mercury and the glass walls of the closed capillary. No difficulty was experienced in sealing in the filament without distorting the adjacent portions of the capillary. In fact, filaments were sealed successfully into capillaries having a bore less than 1 mm. The large bulb (*B*) had a volume of about 60 c.c. and the ratio of the volume of a length of 1 mm. of capillary to that of the bulb was 5.26×10^{-5}. A Gaede mercury pump was used to produce a moderately high vacuum while a liquid-air trap containing cocoanut charcoal, introduced between pump and gauge, was used to attain the highest vacua.

Calibration of Hot-wire Gauge.

Unfortunately, the indications of the hot-wire gauge are dependent upon the character of the gas used. This shortcoming is, however, not a

vital defect, since this gauge may be calibrated against the McLeod gauge itself. The mode of procedure consisted in surrounding the charcoal bulb with liquid air so as to remove all vapors, hydrocarbon, water, etc., from the air which had been admitted into the pump. After measuring the pressure by means of the McLeod gauge, the resultant steady galvanometer deflection was noted (bulb and capillary empty). By carrying out a number of such pairs of observations at different pressures, a calibration curve of the type shown in Fig. 2 was obtained.

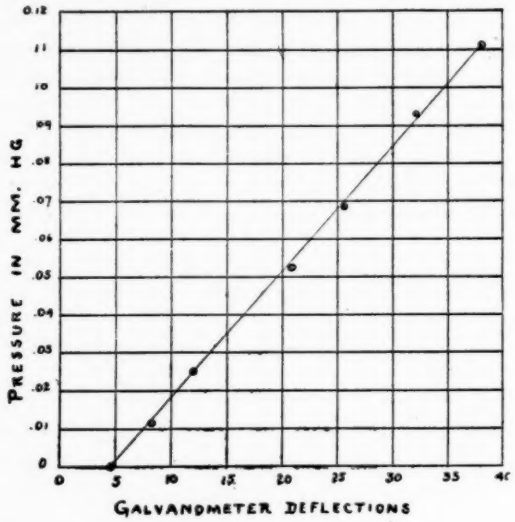

Fig. 2.

It is apparent that the points plotted lie, very approximately, on a straight line. Since mercury vapor is always present, it is evident that only partial pressures, to the exclusion of that of mercury vapor, are being measured.

Reliability and Accuracy.

Upon reducing the pressure below the point at which any difference of level in the two mercury columns L and M (Fig. 1) is discernible, the hot-wire gauge still shows large deflection. As previously stated, this is due to the fact that the original, large volume of gas, after having been forced into the capillary, now exists at a relatively high pressure and hence cools the hot tungsten filament. Since the hot-wire gauge is *used* when the gas is compressed into a limited volume, while it is *calibrated* when capillary and bulb are empty, it is entirely possible that the original calibration does not apply under the conditions attendant upon the actual use of the gauge. To test this point, a certain volume of air was

compressed so that it filled the entire capillary; then, by successive steps, the volume was decreased until the mercury stood 2 mm. below the tip of the tungsten filament. Galvanometer readings, corresponding to these successive volumes were recorded. A characteristic series of results is presented in Table I.

TABLE I.

h	Galvanometer Reading.	P' mm.	P_0 mm.
∞	4.6		
60 mm	7.0	0.0084	2.66×10^{-5}
30	10.0	0.0175	2.76
15	15.0	0.035	2.76
10	19.9	0.051	2.68
8	23.7	0.064	2.70

h = length of air-column in closed capillary.
P' = pressure of gas in closed capillary.
P_0 = pressure in system outside of gauge.

Referring back to the original calibration curve (Fig. 2) the pressures P' in the capillary were evaluated. Since the ratio of a compression is known in each case, it is possible to calculate P_0, the true pressure of the gas in the vacuum system outside the gauge. If the complications, previously referred to, exist, then P_0 might be expected to vary pronouncedly. As a matter of fact P_0 remains nearly constant even though the top of the mercury in the closed capillary is at widely different distances from the filament.

While the constancy of P_0 *points* to the conclusion that the gauge-readings are correct, it does not *establish* the proof. Most fortunately, the McLeod and hot-wire gauges overlapped in a small range of pressures, so that a decisive test was possible. The results are presented in Table II

TABLE II.

	McLeod Gauge.			Hot-wire Gauge.
	h	Δ	P_0	
I	8.0	1. mm.	2.3×10^{-4} mm.	2.0×10^{-4} mm.
II	6.0	0.8	3.35×10^{-4}	4.0×10^{-4} mm.

h = length of trapped air-column in mms.
Δ = difference in level of the two mercury columns in mm.
P_0 = true pressure in mm.

The agreement is well within experimental error which is rather large in the case of the McLeod gauge because of the fact that differences in level (Δ) amounting to a millimeter or less must be read.

An attractive feature of the mode of procedure here presented is the ease with which the hot-wire gauge may be calibrated. It is only necessary to obtain galvanometer deflections (bulb and capillary empty) at two known pressures, preferably as far apart as possible. As a final test, the two pressures were respectively near the upper and lower limits of the McLeod gauge. Upon laying off these two points on coordinate paper, as in Fig. 2, and connecting them by a straight line, a calibration curve was obtained which, when tested for constancy of P_0, showed an average deviation from the mean of only 2.2 per cent.

The range of pressures which may be measured with this arrangement and the magintude of the lowest pressures detectable are rather remarkable. Since the scale divisions on the portable galvanometer were about 3 mm. long, it was easy to read deflections to one-tenth division. Readings were reliable only to .2 divisions so, calculating the galvanometer sensibility on this basis it turned out to be 2.4×10^{-6} amperes. With this combination of gauges as used, it is possible to cover the entire range of pressures extending from 0.275 mm. to 1.7×10^{-7} mm Hg (or 2.25×10^{-4} bars). It is, of course, obvious that the limit of sensibility is determined only by that of the galvanometer. Since the limit of the unaided McLeod gauge is 5.26×10^{-5} mm. it appears that, through the addition of the hot-wire gauge, pressures as low as 1/300th of the above may be measured.

At a time when the true pressure P_0 was 8.4×10^{-7} mm. and the portable galvanometer yielded a deflection of 1 division, a wall galvanometer of sensibility 5×10^{-9} amp. was substituted. As might have been foreseen, a deflection too large to read was obtained. While the calculated sensibility was 3.4×10^{-10} mm. or 4.5×10^{-7} bars it is not feasible to use so sensitive a galvanometer unless the Wheatstone bridge, hot-wire gauge, etc., are surrounded by constant temperature oil-baths, as is customary in bolometer practice.

JOHNS HOPKINS UNIVERSITY,
February, 1921.

NEW BOOKS.

Conrse de Cinematique Theorique. By H. LACAZE. Paris: Gaqthier-Villars et Cie, 1929. Pp. 1 + 138.

The five chapters of this brief treatise deal respectively with:

Vectors; Kintmatics of a point; Movement of a Solid; The Composition of Accelerations; The Displacement of a Vector in a Plane.

More than half of the volume is given to appendices in which the subject matter is developed in somewhat greater detail.

Physical Laboratory Manual. By E. L. CHAFFEE AND F. A. SAUNDERS. Cambridge: Harvard University Press, 1920. Pp. vi + 138.

A set of about fifty well-chosen elementary experiments forming the laboratory portion of an undergraduate course in general physics. The treatment is chiefly quantitative and the distribution of material among the various parts of physics a reasonable one.

Thèses Prèsentees A La Faculté des Sciences de L'Université de Paris. By M. PAUTHENIER. Paris: Masson et Cie, 1920. Pp. 1 + 68.

M. Pauthenier's thesis deals with a new method of producing instantaneous charges, with applications to the Kerr Effect.

The New Physics. By ALBERT C. CREHORE. San Francisco: Journal of Electricity, 1920. Pp. 1 + 111.

This little volume contains a non-mathematical statement of the author's electromagnetic theory of gravitation; a theory which connects the gravitational force directly with the motion of the electrons of matter and which is to be regarded as supplementary to the Einstein theory.

Laboratory Projects in Physics. By FREDERICK F. GOOD. New York: The Macmillan Company, 1920. Pp. xiii + 267.

A school book in which "in accordance with recent tendencies in the teaching of science" numerous familiar household appliances and mechanisms are utilized in the illustration of the principles of physics. The applications are more than commonly sound and successful.

Report of the Chief Signal Officer United States Army to the Secretary of War. Washington: Government Printing Office, 1920. Pp. 1 + 64.

This report deals chiefly with the activities of the U. S. Signal Corps at home and abroad since the signing of the armistice.

Lessons in Mechanics. By WM. S. FRANKLIN AND BARRY MACNUTT. Bethlehem, Pa.: Franklin and Charles, 1919. Pp. xi + 221. Price, $2.00.

Eminent Chemists of our Time. By BENJAMIN HARROW. New York: D. Van Nostrand Company, 1920. Pp. xvl + 248. Price, $2.50 net.

A volume of very readable sketches of Perkin, Mendeléeff, Ramsay, Richards, Van' T Hoff, Arrhenius, Moisson, M. Curie, V. Meyer, Remsen and E. Fischer.

Lessons in Heat. By WM. S. FRANKLIN AND BARRY MACNUTT. Bethlehem, Pa.: Franklin and Charles, 1920. Pp. xi + 147. Price, $2.00.

Lessons in Electricity and Magnetism. By WM. S. FRANKLIN AND BARRY MACNUTT. Bethlehem, Pa.: Franklin and Charles, 1919. Pp. xvi + 254. Price, $2.25.

These three companion volumes are planned for use in elementary undergraduate courses in Physics. They are concise and concrete and conform throughout to the method of treatment characteristic of the authors.